# Appendix 1
S-parameter Basics

A1-1 Definition

The S-parameter (Scattering parameter) expresses device characteristics using the degree of scattering when an AC signal is considered as a wave. The word "scattering" is a general term that refers to reflection back to the source and transmission to other directions. **Figure A1-1** shows an optical analogy. The word "degree" indicates the amount of attenuation or amplification, which is measured using the square root of the electric power.
It is possible to understand all linear characteristics of the device based on the degree of scattering (S-parameter).

Figure A1-1 Conceptual diagram of S-parameter (optical analogy)
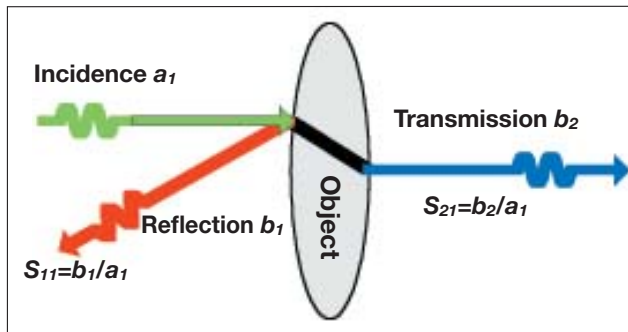
The input and output ports of a device are numbered and the S-parameter that is "Incident at port j→Detected at port i" is described as Sij. Reflection is represented as $i = j$, and transmission is described as $i \neq j$. Therefore, in an n-port device, there are S-parameters of n^2 pieces. When these S-parameters are aligned in a matrix **form (A1-1)**, it is referred to as an S-matrix (Scattering matrix). For a more detailed definition, please refer to the textbooks [1] to [5].

(A1-1)

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & & S_{2n} \\ \vdots & & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

The S-parameter is a ratio, so it is basically a non-dimensional parameter (no unit). However, when describing the magnitude of the S-parameter, the unit "dB" is usually used with a common logarithm. For reference, the following **Table A1-1** shows some representative values.

Table A1-1 Magnitude of the S-parameter

$|S_{ij}|$	$20\log|S_{ij}|$
1	0dB
$1/\sqrt{2}$	-3dB
1/10	-20dB
1/100	-40dB
1/1000	-60dB

A1-2 Characteristics

The following characteristics are very helpful to understand S-parameter concepts.
• If a device is lossless, the S-matrix becomes unitary. Therefore, a lossless 2-port device possesses

$$|S_{11}|^2 + |S_{21}|^2 = 1$$ (Feldtkeller's formula).

There is no loss, so the total amount of the scattering should be 100%. This shows the relationship, "When S_{21} (S_{11}) is large, S_{11} (S_{21}) is small."
• A passive 2-port device possesses

$$|S_{11}|^2 + |S_{21}|^2 \leq 1$$ (The equal sign means lossless as mentioned

above.).

Therefore, S-parameters for a passive device are not over 1 (0dB). The left side of the equation, $PS := |S_{11}|^2 + |S_{21}|^2$, is referred to as the "Power Scattering Ratio".

This ratio shows how much electric power the device consumes. The smaller the result for the calculation ($0 \le PS \le 1$), the greater the loss.

• The cut-off frequency indicates the boundary between the passband and stopband of a device (filter). It can also be referred to as the frequency at which $|S_{ij}|$ = -3dB (half of the electric power passes). Therefore, if a 2-port device is lossless, $|S_{11}|=|S_{21}|$ (this means the intersection point of the $|S_{11}|$ graph and the $|S_{21}|$ graph) at the cut-off frequency (refer to **Figure A1-6** for an example).

• If a device is reciprocal and is not a unidirectional component such as an isolator or circulator, the S-matrix is symmetric. Therefore, $S_{ij} = S_{ji}$.

In mixed-mode S-parameters (refer to A1-10), $S_{cc21}=S_{cc12}$, $S_{cd21}=S_{dc12}$, $S_{dc21}=S_{cd12}$, $S_{dd21}=S_{dd12}$.

A1-3 Touchstone Format

Recently network analyzers are generally used to measure S-parameters. When the data that is to be transferred or that is used for simulation is described numerically, it is convenient to save it as a Touchstone format text file (.s*n*p). **Figure A1-2** shows an example of a Touchstone file.

Figure A1-2 Example of Touchstone file

```
# MHz  S  MA  R 50
0         0.00545    0          0.99455   0          0.99455    0          0.00545   0
0.30000   0.99950   -0.03754    0.00020   89.7291    0.00020    93.8341    0.99971  -0.02354
0.31523   0.99955   -0.03746    0.00020   89.8781    0.00020    89.0219    0.99992  -0.02312
 · · · · ·
6000.00   0.17892   -107.756    0.97259  -16.4840    0.97174   -16.5683    0.18224  -101.842
```

For an explanation of the option line (# row), refer to **Table A1-2** [6]. From the next row, there are numerical values that consist of multiple columns. The far left column indicates frequencies. In this example, they are DC to 6GHz (0Hz to 6000MHz). The frequency range and its interval are not ruled, but it must be arranged in increasing order. The remaining eight columns show the S-parameters at each frequency. The order for a 2-port device is S_{11}, S_{21}, S_{12}, and S_{22}. Each S-parameter is described using two real numbers. (In this example, MA format is used, so the magnitude and phase are real numbers.) Therefore, there are a total of eight columns; nine columns including the frequency. In other cases than a 2-port device, the Touchstone file looks similar, but there are some differences, for example, the numerical values may be ordered in a matrix form.

The description format for other than 2-port devices is almost the same with some minor differences, e.g. the order of the numerical values may be in a matrix form.

Table A1-2 Rules of option line (# row)

Example	Specification
MHz	Unit of frequency
S	Circuit matrix type (such as S/Z/Y)
MA	Expressive form for complex numbers MA: Magnitude / Angle (Phase) RI: Real / Imaginary DB: Magnitude in dB / Angle Angle (phase) unit is degree.
R50	reference impedance/Ω R50 refers to 50Ω

A1-4 Reasons Why S-Parameters are Used

As electronic devices have become faster and faster, greater emphasis has been placed on analog characteristics (SI=Signal Integrity) even in digital circuits, where in the past S-parameters were not so commonly

used. Now, S-parameters are being given more attention, and the following shows why and how S-parameters are used.
• The transfer of electric signals or power (energy) can be expressed by S-parameters, which can show such physical quantities as attenuation of a filter or transducer gain of an active device.
• When the size of a device at high-frequency is similar to the wavelength, it is necessary to consider the time difference for the location. It is easy to understand this phenomenon by using the concept of reflection and transmission. Also, the calculation can be simpler. This is why the S-matrix for a transmission line can be described by the simple formula shown in (**A1-13**).
• A passive device always has S-parameters (not divergence). Therefore, for example, it is valid to analyze an ideal transformer network.
• It is difficult to achieve strict termination conditions such as open and short at high frequencies. Because of resistive termination, S-parameters can be measured at high frequencies. The measured amount makes calculation easier, as there is no need to convert to Z- or Y-parameters.

A1-5 Impedance

The characteristics of a linear 1-port (two-terminal) device can be described by one complex number, such as the impedance, admittance or reflection coefficient. The following gives an overview of these coefficients.

Impedance Z is the ratio of the voltage between terminals and the current through terminal. The inverse of this ratio is admittance Y. These are complex numbers that can be described using two real numbers such as real / imaginary numbers or a polar form.

The real part of the impedance is referred to as "Resistance R" and the imaginary part is referred to as "Reactance X". The real part of the

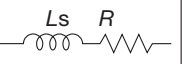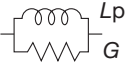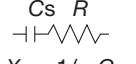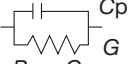admittance is referred to as "Conductance G" and the imaginary part is referred to as "Susceptance B".

These are determined by two real numbers, so it is also possible to describe them in a composition circuit, e.g. capacitor and resistor (C-R) or inductor and resistor (L-R). A parallel or a series connection can be used for composition, so there are a total of four possible composition circuits (refer to **Table A1-3**).

L and C in this circuit are mere parameters that express the imaginary part. For example, the reactance X divided by the angular frequency ω is the series inductance Ls. And the susceptance B divided by the angular frequency ω is the parallel capacitance Cp.

Let's consider whether Cp indicates the capacitance of a capacitor in the following paragraph. The condition that Cp is the true capacitance of a capacitor (proportional to the permittivity of dielectric) is whether the objective capacitor can be expressed approximately using a C-R parallel circuit at that measured frequency. Negative Cp indicates that it is over the self-resonant frequency. When a resistive component is connected in series to a capacitor as in an electrolytic capacitor and the influence of the resistor (cathode) cannot be ignored ($D \gg 1$), it is not possible to understand what the Cp indicates even under the self-resonant frequency. So when we measure the capacitance of an electrolytic capacitor, Cs is usually used. Paradoxically, it is necessary

Table A1-3 Parameters to express impedance
(Impedance $Z=R+jX=|Z|e^{j\theta}$, Admittance $Y=G+jB=|Y|e^{-j\theta}$, and L, C, Q, $D = \tan\delta$)

Domain	Expression Circuit		$D=\tan\delta=1/Q$	Interrelation
Inductive $X\geq0, B\leq0$ $0\leq\theta\leq\pi/2$	Ls R $X=\omega L$s	Lp G $B=-1/\omega L$p	$D=R/X=-G/B$ $D=R/\omega L$s$=\omega L$pG $D=\cot\theta$ $\delta=\pi/2-\theta$	$RG=1/(1+Q^2)$ $XB=-1/(1+D^2)$ Cs$=C$p$(1+D^2)$ Lp$=L$s$(1+D^2)$
Capacitive $X\leq0, B\geq0$ $-\pi/2\leq\theta\leq0$	Cs R $X=-1/\omega C$s	Cp G $B=\omega C$p	$D=-R/X=-G/B$ $D=\omega C$s$R=G/\omega C$p $D=-\cot\theta$ $\delta=\pi/2+\theta$	In case of $D\ll1$ $RG\doteqdot D^2$, $XB\doteqdot-1$ Cs$\doteqdot C$p, Lp$\doteqdot L$s

to determine the measurement mode from either C_p or C_s before measuring. However, if two independent parameters (a complex number) are acquired, it is possible to convert them after measuring by using the formulae shown in **Table A1-3**.

The real part of the impedance and admittance represents device loss. This is sometimes described using the ratio "D" (=tanδ, dissipation factor) between the real and imaginary parts. The δ is equal to the complimentary angle of the phase of either impedance or admittance. However, the value itself is rarely used. If the device is low loss, the tanδ is small, so it is described using a percentage. Also, "Q" (quality factor), the inverse of tanδ is widely used.

When an inductor and a capacitor are connected in series or parallel, it becomes a resonance circuit. The following formula shows the relationship between Q of the resonance circuit (= central frequency divided by half bandwidth, describes the resonance sharpness) and the quality factor of the inductor (Q_L) and capacitor (Q_C) when each has loss.

(A1-2) $\quad \dfrac{1}{Q} = \dfrac{1}{Q_L} + \dfrac{1}{Q_C}$.

Therefore, the Q of inductor / capacitor can be defined as the Q of the resonance circuit when a lossless ($Q = \infty$) capacitor / inductor are connected, respectively.

A1-6 Smith Chart

The S-parameter of a 1-port device is referred to as the reflection coefficient (in other words, S-parameters are the expansion of the reflection coefficient to more than two ports). Reflection coefficient Γ has the following relationship with impedance Z and admittance Y, which is also a 1-port parameter.

(A1-3) $\quad \Gamma = \dfrac{Z - Z_0}{Z + Z_0} = \dfrac{Y_0 - Y}{Y_0 + Y}$.

Where the reference impedance is $Z_0 = 1/Y_0$. The Smith chart has scaling of the complex plane of reflection coefficient which enables impedance to be interpreted directly. As shown in (**A1-3**), if including the infinite point, Γ, Z, and Y correspond one-to-one. **Figure A1-3** shows this bilinear correspondence. This figure indicates that a passive device (right half of the impedance complex plane) locates inside of the Smith chart. Short ($Z = 0\Omega$, the origin of the impedance plane) is the left edge (Γ=-1), open ($Z = \infty\Omega$) is the right edge (Γ=1), and $Z = Z_0$ is at the center ($\Gamma = 0$) in the Smith chart (**Figure A1-4**). The blue line in **Figure A1-4** is "constant Q circle" for $Q = 1$. This is the curve that connects $X = \pm R$ points on the Smith chart, which is the half-line whose slope is ±1 in the impedance plane.

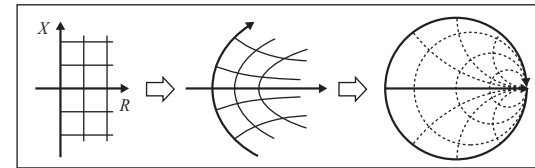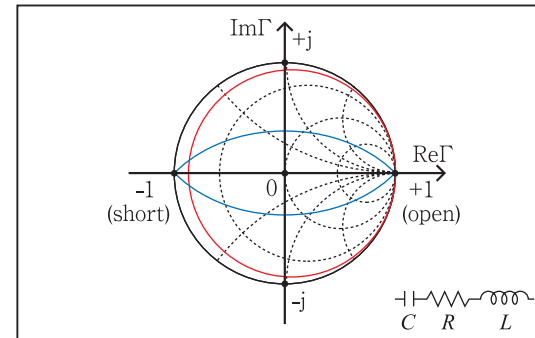Figure A1-3 From impedance plane to Smith chart

Figure A1-4 Trace of series LCR circuit on a Smith chart

As an example, a series LCR resonance circuit is plotted on **Figure A1-4** in red. The frequency is not directly shown in the figure. However, the trace is on the right edge ($\Gamma = 1$, open) in low frequency. From that point, it goes clockwise when the frequency increases and reaches back to the right edge again in a high frequency.
On the impedance plane, the trace is parallel to the imaginary axis through the point $(R, 0)$.
The line is then converted to a circle as shown in **Figure A1-3**.
According to Foster's reactance theorem, reactance is an increasing function of frequency. Therefore, on the Smith chart, the movement is clockwise.
The resonance frequency of the series LCR circuit is given by

(A1-4) $\omega_0 = 1/\sqrt{LC}$.

which is the point when the red line intersects with the real Γ axis. The intersection with both constant Q circles shows the half bandwidth, $\Delta\omega = R/L$ and the Q of the resonance circuit is given by

(A1-5) $Q = R_0/R$.

where $R_0 := \sqrt{L/C}$.

As shown in this figure, the lower the loss (the closer the red line is to the outer circumference of the Smith chart), the narrower the half bandwidth. As a result, Q becomes higher.
The Smith chart is used to measure impedance (converted from the reflection coefficient) and to design matching circuits of microwave amplifiers.

A1-7 The S-parameter for Two-Terminal Components
The following shows how to configure 2-port circuits by arranging a two-terminal component, which is characterized by impedance. **Figure A1-5** shows the two simplest types of circuit. In practical use, most circuits have a combination of these two circuits (refer to the next section for information about connections). In this sense, these two circuits are basic and the S-parameters given in SEAT are taken from these configurations.

Figure A1-5 Configuration of 2-port circuits with a two-terminal component
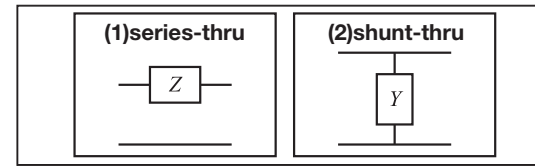
Theoretical formulae for the S-matrix of these configurations are given by

(A1-6) $S_{\text{series-thru}} = \dfrac{1}{Z + 2Z_0} \begin{bmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{bmatrix}$.

(A1-7) $S_{\text{shunt-thru}} = \dfrac{1}{Y + 2Y_0} \begin{bmatrix} -Y & 2Y_0 \\ 2Y_0 & -Y \end{bmatrix}$.

So if the component is lossless, the condition to be cut-off is $Z = j2Z_0$ for series-through and $Y = j2Y_0$ for shunt-through. The following shows an example using ferrite beads (**Figure A1-6**). The cut-off frequency in the low range is around 5MHz. At that frequency, it can be seen that $|Z|=100\Omega$, which is double the reference impedance. If the reference impedance is not 50Ω, the result will be different (refer to **Figure A1-7**).

Figure A1-6 Impedance and S-parameters of ferrite bead
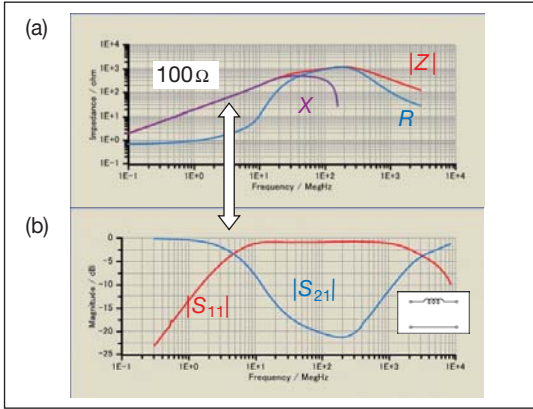
The above shows the ideal relationship between impedance and the S-parameter, but when using actual measurement data, care must be taken. This is because the interaction with the GND exists for actual measurements even when the configuration is the same as **Figure A1-5**.

A1-8 Cascade Connection

Cascade connection is most often used to construct circuits. The S-matrix $\mathbf{S_{MN}}$ for the whole cascade connection of a 2-port device M, and a 2-port device N (the S-matrix for M and N is $\mathbf{S_M}$ and $\mathbf{S_N}$, respectively) is given by

(A1-8) $\mathbf{S_{MN}} = \dfrac{1}{1-S_{M22}S_{N11}} \begin{bmatrix} S_{M11} - S_{N11}|\mathbf{S_M}| & S_{M12}S_{N12} \\ S_{N21}S_{M21} & S_{N22} - S_{M22}|\mathbf{S_N}| \end{bmatrix}$.

Where the reference impedance for the connected ports should be equal to each other, the calculation for cascade connection usually uses an F-or T-matrix. However, the formula (**A1-8**) is convenient when the device is described by using S-parameters and therefore does not require conversion.

The input impedance (described by the reflection coefficient Γ_{IN})

when the port 2 of a device M is terminated can be calculated in case that the device N is a 1-port device (S_{N11} is its reflection coefficient). Therefore, the reflection coefficient Γ_{IN} is the (1,1) element of (**A1-8**),

(A1-9) $\Gamma_{IN} = S_{M11} + \dfrac{S_{M12}S_{N11}S_{M21}}{1 - S_{M22}S_{N11}}$.

A1-9 Characteristics of Transmission Lines

Group delay time and characteristic impedance are the characteristics of a transmission line or a filter, which is considered as a transmission line.

Group delay time t_{GD} using the phase of S_{21} is defined by

(A1-10) $t_{GD} = -\dfrac{\partial}{\partial \omega} \arg S_{21}$.

If this value is not flat to frequency, a signal that contains multiple frequency components such as a digital waveform will be distorted. On the other hand, Characteristic impedance Z_{0t} can be calculated using the Open / Short method ($Z_{0t} = \sqrt{Z_{open} \cdot Z_{short}}$, where Z_{open} and Z_{short} is the input impedance terminated by open and short at the other port, respectively). The input impedance terminated by open or short can be calculated using $S_{N11} = \pm 1$ in (**A1-9**).

As a result, using S-parameters, the characteristic impedance Z_{0t} can be expressed by

(A1-11) $\dfrac{Z_{0t}}{Z_0} = \sqrt{\dfrac{(1+S_{11}+S_{22}+|\mathbf{S}|)(1+S_{11}-S_{22}-|\mathbf{S}|)}{(1-S_{11}+S_{22}-|\mathbf{S}|)(1-S_{11}-S_{22}+|\mathbf{S}|)}}$.

This calculation is the same as image impedance. This formula is based on the Open / Short method, so it is necessary to keep in mind that accurate calculations can only be made when the frequency is lower than a quarter of the wavelength $\lambda/4$. If a coupled line can be decomposed to independent lines by using the adequate

modes (for more information about "mode", please refer to **A1-11**), it is possible to use (**A1-11**) for each mode. In addition, the characteristic impedance can also be measured by TDR (Time Domain Reflectometry) using data dependant on time (location). S-matrix $\mathbf{S_t}$ for a lossless transmission line is given by

(A1-12) $\quad \mathbf{S_t} = \dfrac{1}{1-\rho^2 e^{-j2\theta}} \begin{bmatrix} \rho(1-e^{-j2\theta}) & (1-\rho^2)e^{-j\theta} \\ (1-\rho^2)e^{-j\theta} & \rho(1-e^{-j2\theta}) \end{bmatrix}$.

Where $\rho := (Z_{0t} - Z_0)/(Z_{0t} + Z_0)$, $\theta := \omega t$

and t is the electrical length (Unit: Time). If the characteristic impedance of the transmission line is selected for the reference impedance, then $\rho = 0$, and (**A1-12**) can be simplified to

(A1-13) $\quad \mathbf{S_t} = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{bmatrix}$.

Impedance matching causes no reflection ($S_{11}=0$). The phase of S_{21} is rotated according to the electrical length. If the reference impedance in (**A1-13**) is changed (refer to the next section), it will be returned back to (**A1-12**).

So the denominator in (**A1-12**) indicates multiple reflections due to mismatching.

A1-10 Reference Impedance
Reference impedance is an important concept to understand and use S-parameters [4], [5].
Usually we simply state that S_{21} is xxdB. However, to be exact, it should be stated that S_{21} is xxdB when yyΩ is used as the reference impedance. The reason why the abbreviated expression can be used is that most reference impedance is 50Ω. Still, it is important to keep in mind that the S-parameter is a relative (normalized) value depending on a certain reference value. In other words, reference

impedance is always necessary for acquiring S-parameters (whether by actual measurement or by simulation). Z-parameter (impedance) is usually described with no reference. However, it is possible to describe it using a reference value. For example, when the reference impedance is 50Ω, 200Ω can be described as 4, and 5Ω can be described as 0.1. In this case, the calculation is simple and clear. However, the calculation for the S-parameter is a little more complicated (refer to (**A1-3**)).

50Ω is only a reference value, so it can be changed (renormalization = change of reference impedance). Suppose the S-parameter **S** is already determined for 50Ω, it is possible to transform it to S-parameter **S'** for other reference impedances by using

(A1-14) $\quad \mathbf{S'} = (\mathbf{S} - \rho \mathbf{I})(\mathbf{I} - \rho \mathbf{S})^{-1}$.

Where **I** is the identity matrix,
$\rho := (Z_0' - Z_0)/(Z_0' + Z_0)$, Z_0 and Z_0' are the original and new reference impedance, respectively. It is important to keep in mind that the original S-parameters require all S_{ij} parameters even when you want to know only S'_{21}.

Figure A1-7 Z_0 dependency of the S-parameter of ferrite bead
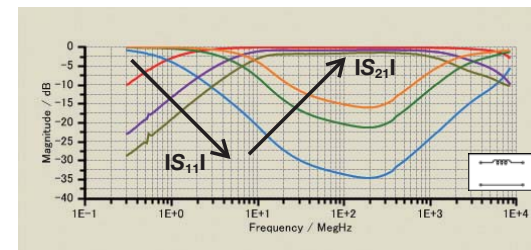
The arrows for $|S_{11}|$ and $|S_{21}|$ indicate the direction of the reference impedance increase (10Ω→50Ω→100Ω)

Figure A1-7 shows S-parameters of ferrite bead with series-through configuration using three reference impedances of 10, 50, and 100Ω. The smaller the reference impedance, the higher the attenuation; shunt-thru capacitors, which are not shown in the figure, give the opposite, namely the bigger the reference impedance, the higher the attenuation.
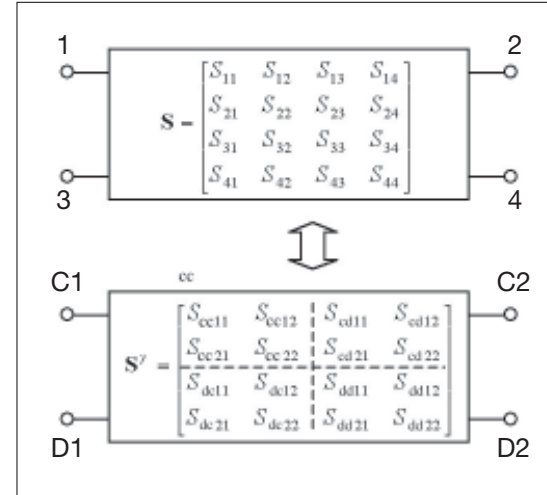
A1-11 Mixed-Mode S-parameters

Since the late 1990s, high-speed differential (or balanced) transmission systems have been developed for practical use in transmitting digital signals. Although this technology has been used for many years, attention has once again been focused on these systems along with high-speed clocks.

A differential transmission system is a system that uses the differential mode (refer to the additional subheading). Therefore, the S-parameter for this system needs to be handled according to the modes, which is referred to as a mixed-mode S-parameter (or modal S-parameter) [5], [7]. Original S-parameter (referred to as a single-ended S-parameter or nodal S-parameter) indicates the response for each port. On the other hand, a mixed-mode S-parameter indicates the response for the sum of two signals (common mode) or the response for the difference of two signals (differential mode). An explanation will be given using a 4-port device that has two input ports and two output ports (**Figure A1-8**). Ports 1 and 3 make up one group, and ports 2 and 4 the other.

Figure A1-8 4-port S-parameters

Mixed-mode S-parameters have the following meanings.
- S_{ccij}: Common mode response.
- S_{ddij}: Differential mode response.
- S_{cdij}, S_{dcij}: Mode conversion between differential mode and common mode.

If the device has good symmetry, the mode conversion should be zero, which means that each mode is independent.

It is possible to calculate mixed-mode S-matrix $\mathbf{S}^\gamma$ by using single-ended S-matrix $\mathbf{S}$ [5],

(A1-15) $\mathbf{S}^\gamma = \mathbf{P}^{-1}\mathbf{SP}.$

Where

(A1-16) $\mathbf{P} := \dfrac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \qquad \mathbf{I} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Here, attention must be paid to the reference impedance.
The reference impedance for the common mode and differential

mode is half and twice the reference impedance for the original single-ended S-parameters, respectively. For example, if the reference impedance for original S-parameters is 50Ω, the reference impedance for the common mode is 25Ω, and for the differential mode is 100Ω. If the other reference impedance is needed, it can be changed by using (**A1-14**).

Figure A1-9 shows an example of actual measurements for a common mode filter (CMF). It can be seen that this CMF attenuates the common mode signal around 100MHz, but transmits the differential mode signal with little loss in that frequency range.

Figure A1-9 Mixed-mode S-parameters of CMF
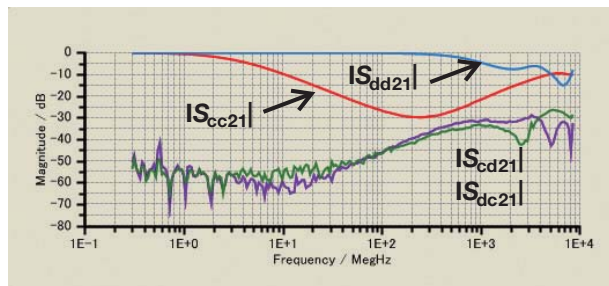
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[6] Agilent Technology, "ADS manual"
Touchstone is the name of a linear circuit simulator by EEsof (currently Agilent Technologies).
[7] David E. BOCKELMAN, William R. EISENSTADT, "Combined differential and common-mode scattering parameters: theory and simulation", IEEE Tarns. MTT, vol. 43, No. 7, pp.1530-1539, 1995 July

Common Mode and Differential Mode

Suppose there are two conductors (and a GND conductor) that are parallel to each other. When the voltage and the current for each conductor are named as V_1, I_1 and V_2, I_2, the common mode voltage Vc and Current Ic and the differential mode voltage Vd and current Id are defined as the following (International electrotechnical vocabulary in EMC, IEC60050-161:1990, JIS C0161: 1997)

· V_c: Average voltage for each conductor
  $$V_c := (V_1 + V_2)/2$$
· I_c: Total current for each conductor
  $$I_c = I_1 + I_2$$
· V_d: Voltage between two conductors
  $$V_d = V_1 - V_2$$
· I_d: Half of the current difference for each conductor
  $$I_d = (I_1 - I_2)/2$$

The common mode indicates the sum of signals, while the differential mode displays the difference between signals. The current in differential mode flows backwards (anti-phase) through two conductors. Therefore, GND is not directly connected (for that reason, it is referred to as normal mode).
On the other hand, the current in the common mode flows in the same direction (in-phase) through two conductors. Consequently, the current flows through the GND conductor (or through another area) and then returns (hence this is referred to as an earth circuit). The common mode is also referred to as asymmetrical (or vertical, as in vertical electrical current) or longitudinal, and the differential mode is also referred to as being symmetrical (or horizontal).