Stieltjes moment sequences of polynomials

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Stieltjes moment sequences

A sequence $(a_n)_{n\geq 0}$ is a **Stieltjes moment sequence** if it has the form

$$a_n = \int_0^\infty x^n d\mu(x)$$

where μ is a nonnegative measure on $[0, \infty)$.

Other Characterizations

(I) A sequence $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if the determinants of the matrices $[a_{i+j}]_{0\leq i,j\leq n}$ and $[a_{i+j+1}]_{0\leq i,j\leq n}$ are positive for all $n \geq 0$

Total Positivity

Let $A = [a_{n,k}]_{n,k \geq 0}$ be a finite or infinite matrix.

We say that A is **totally positive of order r** if all its minors of order $1, 2, \ldots, r$ are nonnegative.

We say that A is **totally positive** if it is totally positive of order r for all $r \geq 1$

A third characterization

Given a sequence $\alpha = (a_n)_{n\geq 0}$, we define the *Hankel matrix of* α, $H(\alpha)$, by

$$H(\alpha) = [a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then α is a Stieltjes moment sequence if and only if $H(\alpha)$ is TP.

Let $\mathbb{R}$ denote the real numbers and $\mathbf{x} = x_1, \ldots, x_n$.

For any polynomial $f(\mathbf{x}) = \sum c_{i_1,\ldots,i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ in $\mathbb{R}[\mathbf{x}]$, we let $f(\mathbf{x})|_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}} = c_{i_1,\ldots,i_n}$ denote the coefficient of $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ in $f(\mathbf{x})$.

We say that $f(\mathbf{x})$ **is x-nonnegative**, written $f(\mathbf{x}) \geq_{\mathbf{x}} 0$, if

$$f(\mathbf{x})|_{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}} \geq 0 \text{ for all } i_1, \ldots, i_n.$$

Let $M = [m_{n,k}(\mathbf{x})]_{n,k \geq 0}$ be a finite or infinite matrix of polynomials in $\mathbb{R}[\mathbf{x}]$.

We say that M is **x-totally positive of order r** ($\mathbf{x}$-TP_r) if all its minors of order $1, 2, \ldots, r$ are polynomials in $\mathbf{x}$ with nonnegative coefficients.

We say M is **x-totally positive** ($\mathbf{x}$-TP) if it is $\mathbf{x}$-totally positive of order r for all $r \geq 1$.

Given a sequence $\alpha = (a_k(\mathbf{x}))_{k\geq 0}$ of polynomials in $\mathbb{R}[\mathbf{x}]$, we define the **Hankel matrix of α**, $H(\alpha, \mathbf{x})$, by

$$H(\alpha, \mathbf{x}) = [a_{i+j}(\mathbf{x})]_{i,j\geq 0} = \begin{bmatrix} a_0(\mathbf{x}) & a_1(\mathbf{x}) & a_2(\mathbf{x}) & a_3(\mathbf{x}) & \cdots \\ a_1(\mathbf{x}) & a_2(\mathbf{x}) & a_3(\mathbf{x}) & a_4(\mathbf{x}) & \cdots \\ a_2(\mathbf{x}) & a_3(\mathbf{x}) & a_4(\mathbf{x}) & a_5(\mathbf{x}) & \cdots \\ a_3(\mathbf{x}) & a_4(\mathbf{x}) & a_5(\mathbf{x}) & a_6(\mathbf{x}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then we say that α is a **Stieltjes moment sequence of polynomials** if and only if $H(\alpha, \mathbf{x})$ is $\mathbf{x}$-TP.

In the case where $n = 1$ so that we are considering polynomials in a single variable, our definition coincides with the definition of Stieltjes moment sequences of polynomials as defined by Wang and Zhu (2016)

Catalan Type numbers (Aigner 1999)

Let $\sigma = (s_k)_{k\geq 0}$ and $\tau = (t_{k+1})_{k\geq 0}$ be two sequences of nonnegative numbers. Then define an infinite lower triangular matrix $A := A^{\sigma,\tau} = [a_{n,k}]_{n,k\geq 0}$ where the $a_{n,k}$s are defined by the recursions

$$a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1} \tag{1}$$

subject to the initial conditions that $a_{0,0} = 1$ and $a_{n,k} = 0$ unless $n \geq k \geq 0$.

Aigner called $A^{\sigma,\tau}$ the **recursive matrix** corresponding to (σ, τ) and he called the sequence $(a_{n,0})_{n\geq 0}$, the **Catalan-like numbers** corresponding to (σ, τ).

Recently, Liang Mu, and Wang (2016) showed that many Catalan-like numbers are Stieltjes moment sequences by proving that the Hankel matrix of the sequence $(a_{n,0})_{n\geq 0}$ is totally positive. Such examples include the Catalan numbers, the Bell numbers, the central Delannoy numbers, the restricted hexagonal numbers, the central binomial coefficients, and the large Schröder numbers.

q-Aigner Sequences (Zhu 2013)

Suppose that we are give three sequence of polynomials over $\mathbb{R}$ with non-negative coefficients

$$\pi = (r_k(q))_{k\geq 1}, \ \sigma = (s_k(q))_{k\geq 0}, \text{ and } \tau = (t_{k+1}(q))_{k\geq 0}.$$

Then we define a lower triangular matrix of polynomials

$$M(q) := M^{\pi,\sigma,\tau}(q) = [m_{n,k}(q)]_{0\leq k\leq n}$$

where the $m_{n,k}(q)$ are defined by the recursions

$$m_{n+1,k}(q) = r_k(q)m_{n,k-1}(q) + s_k(q)m_{n,k}(q) + t_{k+1}(q)m_{n,k+1}(q) \quad (2)$$

subject to the initial conditions that $m_{0,0}(q) = 1$ and $m_{n,k}(q) = 0$

Liu and Wang (2007) defined a sequence of polynomials $(f_n(q))_{n\geq 0}$ over $\mathbb{R}$ to be ***q*-log convex** (q-LCX) if for all $n \geq 1$,

$$(f_n(q))^2 \geq_q f_{n-1}(q) f_{n+1}(q) \tag{3}$$

and defined a sequence of polynomials $(f_n(q))_{n\geq 0}$ to be **strongly *q*-log convex** (q-$SLCX$) if for all $n \geq m \geq 1$,

$$f_n(q) f_m(q) \geq_q f_{n-1}(q) f_{m+1}(q). \tag{4}$$

Theorem 0.1. *Zhu (2013) A sequence of polynomials $(m_{n,0}(q))_{n\geq 0}$ is a q-$SLCX$ sequence of polynomials if for all $k \geq 0$, $s_k(q) s_{k+1}(q) - t_{k+1}(q) r_{k+1}(q) \geq_q 0$.*

Suppose that a and b are nonnegative real numbers and $r_k(q) = 1$ for $k \geq 1$, $s_0(q) = q^2$ and $s_k(q) = 1 + q^2 + a * q^b$ for $k \geq 1$, and $t_1(q) = q^4$ and $t_k(q) = q^2 + q^4$ for $k \geq 2$.

It is easy to check that for all $k \geq 0$, $s_k(q)s_{k+1}(q) - t_{k+1}(q)r_{k+1}(q) \geq_q 0$. First one can compute that

$$\begin{aligned}
m_{0,0}(q) &= 1, \\
m_{1,0}(q) &= q^2, \\
m_{2,0}(q) &= q^4 + 4q^6 + aq^{4+b}, \\
m_{3,0}(q) &= q^4 + 5q^6 + 9q^8 + 2aq^{4+b} + 4aq^{6+b} + a^2q^{4+2b}, \text{ and} \\
m_{4,0}(q) &= q^4 + 8q^6 + 20q^8 + 21q^{10} + 3aq^{4+b} + 13aq^{6+b} + \\
&\quad 15aq^{8+b} + 3a^2q^{4+2b} + 5a^2q^{6+2b} + a^3q^{4+3b}.
\end{aligned}$$

Then one can compute that

$$\det\left(\begin{bmatrix} m_{0,0}(q) & m_{1,0}(q) & m_{2,0}(q) \\ m_{1,0}(q) & m_{2,0}(q) & m_{3,0}(q) \\ m_{2,0}(q) & m_{3,0}(q) & m_{4,0}(q) \end{bmatrix}\right) =$$

$$-q^8 - 4q^{10} + 6q^{12} + 36q^{14} + 27q^{16} - 64q^{18} - 3aq^{8+b} - 2aq^{10+b} + 27aq^{12+b} + 35aq^{14+b} - 48aq^{16+b} - 3a^2q^{8+2b} + 5a^2q^{10+2b} + 14a^2q^{12+2b} - 12a^2q^{14+2b} - a^3q^{8+3b} + 3a^3q^{10+3b} - a^3q^{12+3b}$$

which is not a polynomial in q with nonnegative coefficients for all integers $a, b \geq 0$.

Wang and Zhu (2016) showed that many of the special sequences considered by Zhu are in fact Stieltjes moment sequences of polynomials over q.

(1) The Bell polynomials $B_n(q) = \sum_{k=0}^{n} S(n,k)q^k$ when $r_k(q) = 1$, $s_k(q) = k + q$, and $t_k(q) = kq$. Here $S(n,k)$ is the Stirling number of the second kind which counts the number of set partitions of $\{1, \ldots, n\}$ into k parts.

(2) The Eulerian polynomials $A_n(q) = \sum_{k=0}^{n} A(n,k)q^k$ when $r_k(q) = 1$, $s_k(q) = (k+1)q + k$, and $t_k(q) = k^2 q$. Here $A(n,k)$ is the number of permutations of n with k descents.

(3) The q-Schröder numbers, $r_n(q) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{n+k}{n-k}q^k$ when $r_k(q) = 1$, $s_0(q) = 1 + q$, $s_k(q) = 1 + 2q$ for $k \geq 1$, and $t_k(q) = q(1+q)$.

(4) The q-central Delannoy numbers $D_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k} q^k$ when $r_k(q) = 1$, $s_k(q) = 1 + 2q$, $t_1(q) = 2q(q+1)$, and $t_k(q) = q(1+q)$ for $k > 1$.

(5) The Narayana polynomials $N_n(q) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} q^k$ when $r_k(q) = 1$, $s_0(q) = q$, $s_k(q) = 1 + q$ for $n \geq 1$, and $t_k(q) = q$.

(6) The Narayana polynomials $W_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k$ of type B when $r_k(q) = 1$, $s_k(q) = 1 + q$, $t_1(q) = 2q$, and $t_k(q) = q$ for $k > 1$.

Multivariate Aigner Sequences.

Suppose that we are given three sequences of polynomials over $\mathbb{R}$ with nonnegative coefficients

$$\pi = (r_k(\mathbf{x}))_{k\geq 1},\ \ \sigma = (s_k(\mathbf{x}))_{k\geq 0},\ \text{ and } \tau = (t_{k+1}(\mathbf{x}))_{k\geq 0}.$$

Then we define a lower triangular matrix of polynomials

$$M(\mathbf{x}) := M^{\pi,\sigma,\tau}(\mathbf{x}) = [m_{n,k}(\mathbf{x})]_{0\leq k\leq n}$$

where the $m_{n,k}(\mathbf{x})$ are defined by the recursions

$$m_{n+1,k}(\mathbf{x}) = r_k(\mathbf{x})m_{n,k-1}(\mathbf{x}) + s_k(\mathbf{x})m_{n,k}(\mathbf{x}) + t_{k+1}(\mathbf{x})m_{n,k+1}(\mathbf{x})$$

subject to the initial conditions that $m_{0,0}(\mathbf{x}) = 1$ and $m_{n,k}(\mathbf{x}) = 0$ unless $0 \leq k \leq n$.

Wieghted Motzkin Paths

A Motzkin path is path that starts at $(0,0)$ and consist of three types of steps, up-steps $(1,1)$, down-steps $(1,-1)$, and level-steps $(1,0)$. We let $\mathcal{M}_{n,k}$ denote the set all paths that start at $(0,0)$, end at (n,k), and stays on or above the x-axis.

We weight
an up-step that ends at level k with $r_k(\mathbf{x})$,
a level-step that ends at level k with $s_k(\mathbf{x})$, and
a down-step that ends at level k with $t_{k+1}(\mathbf{x})$.

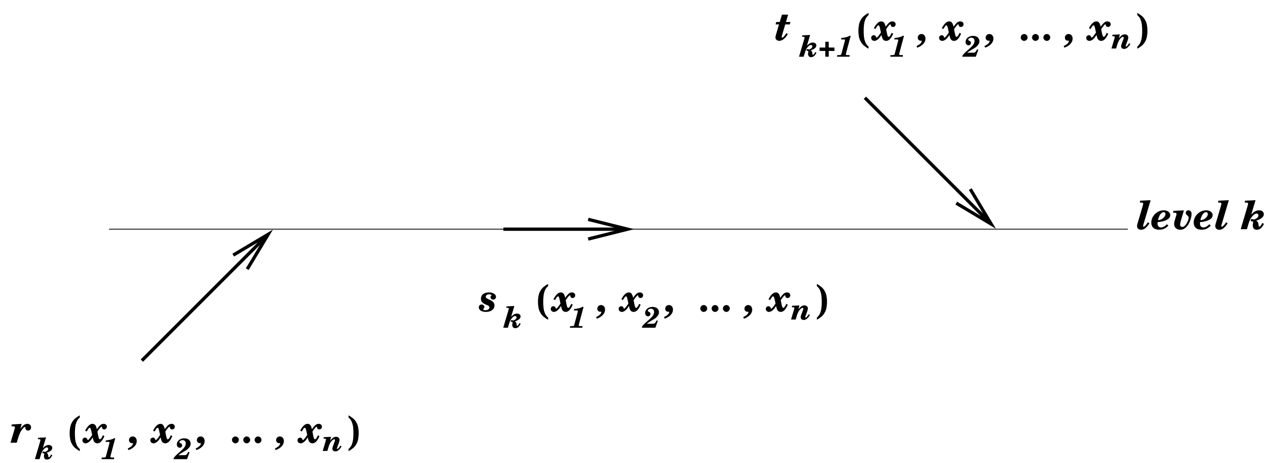

Figure 1: The weight of steps in Motzkin paths

Given a path P in $\mathcal{M}_{n,k}$, we let the weight of P, $w(P)$, equal the product of all the weights of the steps in P. Then if we let

$$m_{n,k}(\mathbf{x}) = \sum_{P \in \mathcal{M}_{n,k}} w(P),$$

it is easy to see that the $m_{n,k}(\mathbf{x})$ satisfy the our recursions

Theorem 0.2. *Let $J = J^{(\pi,\sigma,\tau)}$ be the tridiagonal matrix*

$$J = \begin{bmatrix} s_0(\mathbf{x}) & r_1(\mathbf{x}) & 0 & \ldots & 0 \\ t_1(\mathbf{x}) & s_1(\mathbf{x}) & r_2(\mathbf{x}) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & r_{n-1}(\mathbf{x}) \\ 0 & \ldots & 0 & t_{n-1}(\mathbf{x}) & s_{n-1}(\mathbf{x}) \end{bmatrix}$$

where $\sigma = (s_i(\mathbf{x}))_{i\geq 1}$, $\pi = (r_i(\mathbf{x}))_{i\geq 0}$, and $\tau = (t_{i+1}(\mathbf{x}))_{i\geq 0}$ are sequences of non-zero polynomials over $\mathbb{R}$ with non-negative coefficients. If the coefficient matrix J is $\mathbf{x}$-totally positive, then the $\mathbf{x}$-Catalan-like numbers $m_{n,0}(\mathbf{x})$ corresponding to (π, σ, τ) form a Stieltjes moment sequence of polynomials.

Lemma 0.3. *Suppose that* $A = [a_{i,j}(\mathbf{x})]_{i,j=1,\ldots,n}$ *is triadiagonal matrix of non-negative polynomials in* $\mathbf{x}$ *over* $\mathbb{R}$*. Then* A *is* $\mathbf{x}$*-TP if and only if all of its consecutive principle minors are polynomials in* $\mathbf{x}$ *with non-negative coefficients.*

Lemma 0.4. *Let $J = J^{(\pi,\sigma,\tau)}$ be the tridiagonal matrix*

$$J = \begin{bmatrix} s_0(\mathbf{x}) & r_1(\mathbf{x}) & 0 & \dots & 0 \\ t_1(\mathbf{x}) & s_1(\mathbf{x}) & r_2(\mathbf{x}) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & r_{n-1}(\mathbf{x}) \\ 0 & \dots & 0 & t_{n-1}(\mathbf{x}) & s_{n-1}(\mathbf{x}) \end{bmatrix}$$

where $\sigma = (s_i(\mathbf{x}))_{i\geq 1}$, $\pi = (r_i(\mathbf{x}))_{i\geq 0}$, and $\tau = (t_{i+1}(\mathbf{x}))_{i\geq 0}$ are sequences of non-zero polynomials over $\mathbb{R}$ with non-negative coefficients such that

1. $s_0(\mathbf{x}) \geq 1$,
2. $s_i(\mathbf{x})s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x})r_{i+1}(\mathbf{x}) \geq_{\mathbf{x}} 0$ *for all* $i \geq 0$,
3. $s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x})r_{i+1}(\mathbf{x}) \geq_{\mathbf{x}} 0$ *for all* $i \geq 0$, *and*
4. $s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x})r_{i+1}(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$ *for all* $i \geq 0$.

Then A *is* $\mathbf{x}$*-TP.*

Lemma 0.5. *Let*

$$(b_1(x_1,\ldots,x_n), b_2(x_1,\ldots,x_n),\ldots) \text{ and}$$
$$(c_1(x_1,\ldots,x_n), c_2(x_1,\ldots,x_n),\ldots)$$

be sequences of polynomials in $\mathbf{x} = (x_1,\ldots,x_n)$ *with non-negative coefficient over* $\mathbf{R}$*. Then the tridiagonal matrix*

$$J^{b,c} = \begin{bmatrix} b_1(\mathbf{x})+c_1(\mathbf{x}) & 1 & & \\ b_2(\mathbf{x})c_1(\mathbf{x}) & b_2(\mathbf{x})+c_2(\mathbf{x}) & 1 & \\ & b_3(\mathbf{x})c_2(\mathbf{x}) & b_3(\mathbf{x})+c_3(\mathbf{x}) & \ddots \\ & \ddots & \ddots & \end{bmatrix}$$

is $\mathbf{x}$*-TP.*

Given a polynomial $a(x_1, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in I} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$ where I is finite index set and $c_{i_1,\ldots,i_n} \neq 0$ for all $(i_1, \ldots, i_n) \in I$, we let the degree of $a(x_1, \ldots, x_n)$, $\deg(a(x_1, \ldots, a_n))$, equal $\max(\{i_1 + \cdots + i_n : (i_1, \ldots, i_n) \in I\}$. We say that $a(x_1, \ldots, x_n)$ is **homogeneous if of degree** n if $i_1 + \cdots + i_n = n$ for all $(i_1, \ldots, i_n) \in I$ and is inhomogeneous otherwise. If $a(x_1, \ldots, x_n)$ had degree n, then we let

$$H_{x_0}(a(x_1, \ldots, x_n)) = x_0^n a\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

For example, if $a(x_1, x_2) = 1 + x_1 + x_1x_2 + x_1^3$, then

$$H_{x_0}(a(x_1, x_2)) = x_0^3\left(1 + \frac{x_1}{x_0} + \frac{x_1}{x_0}\frac{x_2}{x_0} + \frac{x_1}{x_0}\frac{x_1}{x_0}\frac{x_1}{x_0}\right) = x_0^3 + x_0x_1x_2 + x_1^3.$$

Clearly for any polynomial $a(x_1, \ldots, x_n)$ had degree n, $H_{x_0}(a(x_1, \ldots, x_n))$ is a homogeneous polynomial.

Theorem 0.6. *Suppose that* $\alpha = (a_0(x_1, \ldots, x_n), a_1(x_1, \ldots, x_n), a_2(x_1, \ldots, x_n), \ldots)$ *is a Stieltjes moment sequence of polynomials such that for all* $n \geq 0$, $\deg(a_n(x_1, \ldots, x_n)) = n$. *Then* $H_{x_0}(\alpha) = (H_{x_0}(a_0(x_1, \ldots, x_n)), H_{x_0}(a_1(x_1, \ldots, x_n)), H_{x_0}(a_2(x_1, \ldots, x_n)), \ldots)$ *is a Stieltjes moment sequence of polynomials.*

Example 1.
Let $\pi = (r_1(q), r_2(q), r_3(q), \ldots) = (1, 1, 1, \ldots)$,
$\sigma = (s_0(q), s_1(q), s_2(q), \ldots) = (1, 1+q, 1+q, \ldots)$ and
$\tau = (t_1(q), t_2(q), t_3(q), \ldots) = (q, q, q, \ldots)$. It is easy to check that these sequences satisfy the hypothesis of Lemma 0.4.

$$
\begin{aligned}
a_{0,0}(q) &= 1, \\
a_{n+1,0}(q) &= a_{n,0}(q) + q a_{n,1}(q) \text{ for } n \geq 1, \text{ and} \\
a_{n+1,k}(q) &= a_{n,k-1}(q) + (1+q) a_{n,k}(q) + q a_{n,k+1}(q) \text{ for } 1 \leq k \leq n.
\end{aligned}
$$

where $a_{n,k}(q) = 0$ unless $n \geq k \geq 0$.

$a_{n,k}(q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k) where the weights of up-steps are 1, the weights of down-steps are q and the weights of level-steps are 1 at level 0 and $1+q$ at levels $k > 0$.

For example, if $A(q) = [a_{n,k}(q)]$, then

$$A(q) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1+q & 2+q & 1 & & \\ 1+3q+q^2 & 3+5q+q^2 & 3+2q & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 0.7. *The sequence $(a_{n,0}(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

Riordan Arrays

A **Riordan array**, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the kth column is $x^k h^k(x) d(x)$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1$ and $h(0) \neq 0$

A Riordan array $R = [r_{n,k}]_{n,k\geq 0}$ can be characterized by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ such that for $n, k \geq 0$,

$$r_{0,0} = 1, \quad r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j}, \quad r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n,k+j}. \tag{5}$$

Call $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ the A- and Z-sequences of R respectively.

Let $Z(x) = \sum_{n\geq 0} z_n x^n$ and $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating functions of $(z_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ respectively.

Then

$$d(x) = \frac{1}{1 - xZ(xh(x))}, \quad h(x) = A(xh(x)). \tag{6}$$

The recursive matrix $R(a,b;c,e) = [r_{n,k}]_{n,k\geq 0}$ defined by

$$\begin{cases} r_{0,0} = 1, \quad r_{n+1,0} = ar_{n,0} + br_{n,1}, \\ r_{n+1,k+1} = r_{n,k} + cr_{n,k+1} + er_{n,k+2}. \end{cases} \tag{7}$$

The coefficient matrix of (7) is

$$J(p,q;s,t) = \begin{bmatrix} a & 1 & & & \\ b & c & 1 & & \\ & e & c & 1 & \\ & & e & c & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{8}$$

Then $R(a, b; c, e)$ is a Riordan array with $Z(x) = a + bx$ and $A(x) = 1 + cx + ex^2$. Let $R(a, b; c, e) = (d(x), h(x))$. Then by (6), we have

$$d(x) = \frac{1}{1 - x(a + bxh(x))}, \quad h(x) = 1 + cxh(x) + ex^2h^2(x).$$

It follows that

$$h(x) = \frac{1 - cx - \sqrt{1 - 2cx + (c^2 - 4e)x^2}}{2ex^2}$$

and

$$d(x) = \frac{2e}{2e - b + (bc - 2ae)x + b\sqrt{1 - 2cx + (c^2 - 4e)x^2}}.$$

Taking $a = 1$, $b = q$, $c = 1 + q$ and $e = q$ in (8), we obtain the generating function of $a_{n,0}(q)$ is

$$d(x, q) = \frac{2}{1 + (q - 1)x + \sqrt{1 - 2(1 + q)x + (1 - q)^2x^2}}.$$

(1) When we set $q = 1$ in $A(q)$, we obtain the Catalan triangle of Aigner, OEIS [A039599]. $a_{n,0}(1) = C_n$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. It follows that $a_{n,0}(q)$ is a q-analogue of the Catalan number C_n.

(2) When we set $q = 2$ in $A(q)$, we obtain the triangle [A172094] and $a_{n,0}(2)$ are the little Schröder numbers S_n. It follows that $a_{n,0}(2q)$ is a q-analogue of n-th little Schröder number S_n.

(3) When we set $q = 3$, the sequence $(a_{n,0}(3))_{n\geq 0}$ is sequence [A007564]. It follows that $a_{n,0}(3q)$ is a q-analogue of the sequence [A007564].

(4) When we set $q = 4$, the sequence $(a_{n,0}(4))_{n\geq 0}$ is sequence [A059231]. It follows that $a_{n,0}(4q)$ is a q-analogue of the sequence [A059231].

Example 2.

Let $\pi = (r_1(q), r_2(q), r_3(q), \ldots) = (1, 1, 1, \ldots)$,
$\sigma = (s_0(q), s_1(q), s_2(q), \ldots) = (1+q+q^2, 1+q+q^2, 1+q+q^2, \ldots)$
and $\tau = (t_1(q), t_2(q), t_3(q), \ldots) = (q, q, q, \ldots)$. It is easy to check that these sequences satisfy the hypothesis of Lemma 0.4.

$$
\begin{aligned}
d_{0,0}(q) &= 1, \\
d_{n+1,0}(q) &= (1+q+q^2)d_{n,0}(q) + qd_{n,1}(q) \text{ for } n \geq 1, \text{and} \\
d_{n+1,k}(q) &= d_{n,k-1}(q) + (1+q+q^2)d_{n,k}(q) + qd_{n,k+1}(q) \text{ for } 1 \leq k \leq n
\end{aligned}
$$

where $d_{n,k}(q) = 0$ unless $n \geq k \geq 0$.

$d_{n,k}(q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k) where the weights of up-steps are 1, the weights of down-steps are q, and the weights of level-steps $1+q+q^2$.

$D(q) = [d_{n,k}(q)]$, then

$$D(q) = \begin{bmatrix} 1 & & & & \\ 1+q+q^2 & 1 & & & \\ 1+3q+3q^2+2q^3+q^4 & 2+2q+2q^2 & 1 & & \\ \begin{array}{c}\big(1+6q+9q^2+10q^3+ \\ 6q^4+3q^5+q^6\big)\end{array} & 3+8q+9q^2+6q^3+3q^4 & 3+3q+3q^2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 0.8. *The sequence $d_{n,0}(q)$ is a Stieltjes moment sequence of polynomials.*

Taking $a = 1 + q + q^2$, $b = q$, $c = 1 + q + q^2$ and $e = q$ in (8), we obtain the generating function of $d_{n,0}(q)$ is

$$d(x, q) = \frac{2}{1 - (1 + q + q^2)x + \sqrt{1 - 2(1 + q + q^2)x + ((1 + q + q^2)^2 - 4q)x^2}}.$$

In this case, the triangle $D(1)$ is [A091965] and the first column $(d_{n,0}(1))_{n\geq 0}$ is sequence [A002212]. $d_{n,0}(1)$ counts the number of 3-color Motzkin paths of length n and the number of restricted hexagonal polynomials with n cells.

Example 3.

Let $\pi = (r_1(p,q), r_2(p,q), r_3(p,q), \ldots) = (1, 1, 1, \ldots)$, $\sigma = (s_0(p,q), s_1(p,q), s_2(p,q), \ldots) = (1+p+q, 1+p+q, 1+p+q, \ldots)$ and $\tau = (t_1(p,q), t_2(p,q), t_3(p,q), \ldots) = (q, q, q, \ldots)$. It is easy to check that these sequences satisfy the hypothesis of Lemma 0.4.

$$\begin{aligned} c_{0,0}(p,q) &= 1, \\ c_{n+1,0}(p,q) &= (1+p+q)c_{n,0}(p,q) + qc_{n,1}(p,q) \text{ for } n \geq 1, \text{ and} \\ c_{n+1,k}(p,q) &= c_{n,k-1}(p,q) + (1+p+q)c_{n,k}(p,q) + qc_{n,k+1}(p,q) \text{ for } 1 \leq k \leq n \end{aligned}$$

where $c_{n,k}(p,q) = 0$ unless $n \geq k \geq 0$.

$c_{n,k}(p,q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k) where the weights of up-steps are 1, the weights of down-steps are q, and the weights of level-steps $1+p+q$.

For example, if $C(p,q) = [c_{n,k}(p,q)]$, then

$$C(p,q) = \begin{bmatrix} 1 & & & & \\ 1+p+q & 1 & & & \\ 1+3p+2q+2pq+p^2+q^2 & 2+2p+2q & 1 & & \\ \begin{matrix}\big(1+6p+3q+6p^2+9pq+3q^2+ \\ p^3+3p^2q+3pq^2+q^3\big)\end{matrix} & \begin{matrix}(3+8p+6q+ \\ 3p^2+6pq+3q^2\big)\end{matrix} & 3+3p+3q & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 0.9. *The sequence $(c_{n,0}(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

Taking $a = 1+p+q$, $b = q$, $c = 1+p+q$ and $e = q$ in (8), we obtain the generating function of $c_{n,0}(p,q)$ is

$$d(x,p,q) = \frac{2}{1-(1+p+q)x+\sqrt{1-2(1+p+q)x+((1+p+q)^2-4q)x^2}}.$$

1. When we set $p = q = 1$ in $(c_{n,0}(1,1))_{n\geq 0}$, we obtain the $1, 3, 10, 36, 137, \ldots$ which is sequence [A002212]. Besides counting 3-colored Motzkin path, it also the number of restricted hexagonal polynomials with n-cells.

2. When we set $p = 1$ and $q = 2$ in $(c_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 4, 18, 88, 456, 2464, \ldots$ which is sequence [A024175].

3. When we set $p = 2$ and $q = 2$ in $(c_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 4, 20, 112, 672, 4224, \ldots$ which is sequence [A003645] whose n-th term is $2^n C_{n+1}$.

Variations of Example 3

Define $t_k^{(s)}(p,q)$ to be q if $k \le s$ and p if $k > s$ and let
$\tau^{(s)} = (t_1^{(s)}(p,q), t_2^{(s)}(p,q), t_3^{(s)}(p,q), \ldots)$.
It is easy to see the sequences
$\pi = (r_1(p,q), r_2(p,q), r_3(p,q), \ldots) = (1, 1, 1, \ldots)$,
$\sigma = (s_0(p,q), s_1(p,q), s_2(p,q), \ldots) = (1+p+q, 1+p+q, 1+p+q, \ldots)$
and $\tau^{(s)}$ satisfy the hypothesis of Lemma 0.4 for all s.

Then we can define the polynomials $c_{n,k}^{(s)}(p,q)$ by

$$\begin{aligned} c_{0,0}^{(s)}(p,q) &= 1, \\ c_{n+1,0}^{(s)}(p,q) &= (1+p+q)c_{n,0}^{(s)}(p,q) + t_1^{(s)}(p,q)c_{n,1}^{(s)}(p,q) \text{ for } n \geq 1, \text{ and} \\ c_{n+1,k}^{(s)}(p,q) &= c_{n,k-1}^{(s)}(p,q) + (1+p+q)c_{n,k}^{(s)}(p,q) + t_{k+1}^{(s)}(p,q)c_{n,k+1}^{(s)}(p,q) \end{aligned}$$

for $1 \leq k \leq n$ where $c_{n,k}^{(s)}(p,q) = 0$ unless $n \geq k \geq 0$.

Theorem 0.10. *For all $s \geq 0$, $(c_{n,0}^{(s)}(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

One of the advantages of this set up is that we can set $p = 0$ in such sequences. In particular, $(c^{(s)}_{n,0}(0, q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials. In such a situation, $c^{(s)}_{n,0}(0, q)$ is the sum over the weights of 2-colored Motzkin paths of height $\leq s$. That is, the level steps can be colored with color 0 which has weight 1 or colored with color 1 which has weight q. The down-steps all have weight q and the up-steps all have weight 1.

Multivariate variations of Example 3.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ where $n \geq 3$ and let $1 \leq s_1 < \cdots < s_{n-1}$.
Then let $r_i(\mathbf{x}) = 1$ for all $i \geq 1$,
$s_i(\mathbf{x}) = 1 + x_1 + \cdots + x_n$ for all $i \geq 1$, and
$t_i^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$ equal x_1 if $i \leq s_1$, x_j if $s_{j-1} < i \leq s_j$, and x_n if $i > s_{n-1}$.

Then let $\pi = (r_1(\mathbf{x}), r_2(\mathbf{x}), r_3(\mathbf{x}), \ldots) = (1, 1, 1, \ldots)$,
$\sigma = (s_0(\mathbf{x}), s_1(\mathbf{x}), s_2(\mathbf{x}), \ldots)$ and
$\tau^{(s_1,\ldots,s_{n-1})} = (t_1^{(s_1,\ldots,s_{n-1})}(\mathbf{x}), t_2^{(s_1,\ldots,s_{n-1})}(\mathbf{x}), t_3^{(s_1,\ldots,s_{n-1})}(\mathbf{x}), \ldots)$. It is easy to check that for any $1 \leq s_1 < \cdots < s_{n-1}$, these sequences satisfy the hypothesis of Lemma 0.4.

In this case, we are considering the polynomials defined by

$$c_{0,0}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) = 1,$$

$$c_{n+1,0}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) =$$
$$(1 + x_1 + \cdots + x_n)c_{n,0}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) + t_1^{(s_1,\dots,s_{n-1})}(\mathbf{x})c_{n,1}^{(s_1,\dots,s_{n-1})}(\mathbf{x})$$
for $n \geq 1$, and

$$c_{n+1,k}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) =$$
$$c_{n,k-1}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) + (1 + x_1 + \cdots + x_n)c_{n,k}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) +$$
$$t_{k+1}^{(s_1,\dots,s_{n-1})}c_{n,k+1}^{(s_1,\dots,s_{n-1})}(\mathbf{x})$$
for $1 \leq k \leq n$,

where $c_{n,k}^{(s_1,\dots,s_{n-1})}(\mathbf{x}) = 0$ unless $n \geq k \geq 0$.

$c_{n,k}(\mathbf{x})$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). where the weights of up-steps are 1, the weights of down-steps ending at level k are $t_{k+1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$, and the weights of level-steps $1+x_1+\cdots+x_n$.

In particular, we can interpret $c_{n,0}(\mathbf{x})$ as weighted sum over $n+1$-colored Motzkin paths. That is, the levels of the Motzkin path can be colored with one of n+1 colors, namely, color 0 which has weight 1, color i which has weight x_i for $i = 1, \ldots, n$, and the down-steps that end at level k have weight $t_{k+1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$.

Theorem 0.11. *For all* $1 \leq s_1 < \cdots < s_{n-1}$, $(c_{n,0}^{(s_1,\ldots,s_{n-1})}(x_1,\ldots,x_n))_{n\geq 0}$ *is a Stieltjes moment sequence of polynomials.*

Example 4.

Let $\pi = (r_1(p,q,r), r_2(p,q,r), r_3(p,q,r), \ldots) = (1,1,1,\ldots)$, $\sigma = (s_0(p,q,r), s_1(p,q,r), s_2(p,q,r), \ldots) = (q+r, p+q+r, p+q+r, \ldots)$ and $\tau = (t_1(p,q,r), t_2(p,q,r), t_3(p,q,r), \ldots) = (q(p+r), q(p+r), q(p+r), \ldots)$.

$$\begin{aligned}
i_{0,0}(p,q,r) &= 1, \\
i_{n+1,0}(p,q,r) &= (q+r)i_{n,0}(p,q,r) + q(p+r)i_{n,1}(p,q,r) \text{ for } n \geq 1, \text{and} \\
i_{n+1,k}(p,q,r) &= i_{n,k-1}(p,q,r) + (p+q+r)i_{n,k}(p,q,r) + q(p+r)i_{n,k+1}(p,q,r)
\end{aligned}$$

for $1 \leq k \leq n$ where $i_{n,k}(p,q,r) = 0$ unless $n \geq k \geq 0$.

We can interpret $i_{n,k}(p,q,r)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k) where the weights of up-steps are 1, the weights of the down-steps are $q(p+r)$, and the weights of the level steps at level 0 is $q+r$ and the weights of the level steps at level $k \geq 1$ are $p+q+r$.

For example, if $I(p,q,r)=[i_{n,k}(q)]$, then

$$I(p,q,r) =$$

$$\left[\begin{array}{ccccc}
1 & & & & \\
q+r & 1 & & & \\
(pq+q^2)+3qr+r^2 & (p+2q)+2r & 1 & & \\
\begin{array}{c}\big(p^2q+3pq^2+q^3\big)+\\(4pq+6q^2)r+6qr^2+r^3\end{array} & (p^2+5pq+3q^2)+(3p+8q)r+3r^2 & (2p+3q)+3r & 1 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right].$$

Theorem 0.12. *The sequence $(i_{n,0}(p,q,r))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

(1) $i_{n,0}(p,q,0) = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} q^k p^{n-k}$ from which it follows that $i_{n,0}(1,1,0) = C_n$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number.

(2) $i_{n,0}(p,q,1) = \sum_{k=1}^{n} \frac{1}{k+1}\binom{n+k}{k}\binom{n}{k} q^k p^{n-k}$ from which it follows that $(i_{n,0}(1,1,1))_{n\geq 0}$ is sequence [A006318] which is the sequence of large Schöoder numbers.

(3) We can show that $(i_{n,0}(1,1,r))_{n\geq 0}$ is the triangle [A060693].

(4) The sequence $(i_{n,0}(1,1,2))_{n\geq 0}$ starts out $1, 3, 12, 57, 300, 1686, 9912, \ldots$ which is sequence [A047891].

(5) The sequence $(i_{n,0}(1,2,1))_{n\geq 0}$ starts out $1, 3, 13, 67, 381, 2307, 14598, \ldots$ which is sequence [A064062] of the generalized Catalan numbers.

(6) The sequence $(i_{n,0}(2,1,1))_{n\geq 0}$ starts out
$1, 2, 7, 32, 166, 926, 5419, 32816, \ldots$ which is sequence [A108524].

(7) The sequence $(i_{n,0}(2,2,1))_{n\geq 0}$ starts out
$1, 3, 15, 93, 645, 4791, 37275, \ldots$ which is sequence [A103210].

(8) The sequence $(i_{n,0}(2,1,2))_{n\geq 0}$ starts out
$1, 3, 13, 71, 441, 2955, 20805, \ldots$ which is sequence [A162326].

(9) The sequence $(i_{n,0}(1,2,2))_{n\geq 0}$ starts out
$1, 2, 7, 32, 166, 926, 5419, 32816, \ldots$ which is sequence [A243626].

(10) The sequence $(i_{n,0}(1,1,3))_{n\geq 0}$ starts out
$1, 4, 20, 116, 740, 5028, 35700, \ldots$ which is sequence [A082298].

(11) The sequence $(i_{n,0}(1,3,1))_{n\geq 0}$ starts out
$1, 4, 22142, 1006, 7570, 59410, \ldots$ is sequence [A243626].

(12) The sequence $(i_{n,0}(3,1,1))_{n\geq 0}$ starts out
$1, 2, 8, 44, 276, 1860, 13140, \ldots$ does not appear in the OEIS.

Example 5

For any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $(des)(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$ and $(ris)(\sigma) = |\{i : \sigma_i < \sigma_{i+1}\}|$. Wang and Zhu (2016) proved that $(E_n(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where

$$E_n(q) = \sum_{k=1}^{n} E_{n,k} q^k = \sum_{\sigma \in S_n} q^{\mathrm{des}(\sigma)}.$$

It follows from Theorem 0.6 that $(E_n(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where

$$E_n(p,q) = \sum_{k=1}^{n} E_{n,k} q^k p^{n-k} = \sum_{\sigma \in S_n} q^{\mathrm{des}(\sigma)} p^{\mathrm{ris}(\sigma)+1}.$$

Let $(b_1(p,q), b_2(p,q), b_3(p,q), \ldots) = (p, 2p, 3p, \ldots)$ and $(c_1(p,q), c_2(p,q), c_3(p,q), \ldots) = (0, q, 2q, \ldots)$. Using Lemma 0.5, we see that $J^{b,c} := J^{\pi,\sigma,\tau}$ where
$\pi = (r_1(p,q), r_2(p,q), r_3(p,q), \ldots) = (1, 1, 1, \ldots)$,
$\sigma = (s_0(p,q), s_1(p,q), s_2(p,q), \ldots) = (p, 2p+q, 3p+2q, \ldots)$ and
$\tau = (t_1(p,q), t_2(p,q), t_3(p,q), \ldots) = (pq, 2^2pq, 3^2pq, \ldots)$.

$$\begin{aligned} h_{0,0}(p,q) &= 1, \\ h_{n+1,0}(p,q) &= ph_{n,0}(p,q) + pqh_{n,1}(p,q) \text{ for } n \geq 1, \text{and} \\ h_{n+1,k}(p,q) &= h_{n,k-1}(p,q) + ((k+1)p + kq)h_{n,k}(p,q) + (k+1)^2pqh_{n,k+1}(p,q) \end{aligned}$$

for $1 \leq k \leq n$ where $h_{n,k}(p,q) = 0$ unless $n \geq k \geq 0$.
$h_{n,k}(p,q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k) where the weights of up-steps are 1, the weights of down-steps that ends at level k is $(k+1)^2pq$, and the weights of the level steps at level k are $(k+1)p + kq$.

One can show that $h_{n,0}(p,q) = E_n(p,q)$.

Theorem 0.13. *The sequence* $\left(\sum_{\sigma\in S_n} p^{\mathrm{ris}(\sigma)+1} q^{\mathrm{des}(\sigma)}\right)_{n\geq 0}$ *is a Stieltjes moment sequence of polynomials.*