Some inequalities on singular values and eigenvalues arising from Hua's determinantal inequality and Ando's Young inequality

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- Definitions and notations

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- Generalization of Hua's determinantal inequality

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- $A \in \mathbb{M}_n$ is called *contractive* if $\sigma_1(A) \leq 1$, equivalently, $I_n \geq A^*A$.

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Lin proves that for contractive $A,\ B \in M_n$,

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As a complement to (2), Lin proves that for contractive $A,\ B \in M_n$,

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Let $A, B \in \mathbb{M}_n$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, there is a unitary matrix U such that

$$U|AB^*|U^* \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q. \tag{4}$$

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The problem remains open for $p \neq 2$.

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Lin conjectured that (3) holds with AB^* is replaced by A^*B.

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Let $A, B \in \mathbb{M}_n$ be contractive. Then

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$$U|A_1 A_2^* A_3|U^* \le \frac{|A_1|^{p_1}}{p_1} + \frac{|A_2|^{p_2}}{p_2} + \frac{|A_3|^{p_3}}{p_3}$$

may not hold for any unitary U.

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Suppose $A_1, \ldots, A_m \in \mathbb{M}_n$ and $p_1, \ldots, p_m > 0$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1.$

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Suppose $A_1, \ldots, A_m \in \mathbb{M}_n$ and $p_1, \ldots, p_m > 0$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$. Let $\tilde{A}_j \in \{A_j,\ A_j^*\}$ for $1 \leq j \leq m$. There exist unitaries $U_1, \ldots, U_m$ such that

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Generalization of Hua's determinantal inequality

Suppose $A_1, \ldots, A_m \in \mathbb{M}_n$ are contractive matrices, $r \geq 1$ and $p_1, \ldots, p_m > 0$ satisfy $\dfrac{1}{p_1} + \cdots + \dfrac{1}{p_m} = 1$.

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$$\prod_{j=1}^{k} (1 - \lambda_j(|A_1 \cdots A_m|)^r) \geq \prod_{i=1}^{m} \prod_{j=1}^{k} (1 - \lambda_j(|A_i|)^{rp_i})^{\frac{1}{p_i}} .$$