Some Optimization Problems in Quantum Information Science

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- In quantum science, one needs to manipulate quantum states using quantum operations.
- One may also want to estimate the change of a quantum states after they go through a certain quantum channel.

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Example Suppose $\mathcal{F}_1, \mathcal{F}_2 \subseteq M_2$. Then ...

Some Known Results

- (Chefles, Jozsa, Winter, 2004) $\mathcal{F}_1, \mathcal{F}_2$ are families of pure states

  $$\rho_i = x_i x_i^* \text{ and } \sigma_i = y_i y_i^* \text{ for } i = 1, \dots, k.$$

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$$F(\rho_1, \rho_2) = \|\sqrt{\rho_1}\sqrt{\rho_2)}\| \leq \|\sqrt{\sigma_1}\sqrt{\sigma_2}\|. \quad (1)$$

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Question Can we find a more explicit (and symmetric) conditions on x_1, x_2, x_3, and $\sigma_1, \sigma_2, \sigma_3$ for the existence of Φ?

A Numerical Scheme

- (Choi, 1975) A linear operator $\Phi : M_n \to M_m$ is a quantum operation if and only if the (Choi) matrix $P = (\Phi(E_{ij}))_{1 \le i,j \le n} \in M_n(M_m)$ is positive semi-definite with $\mathrm{tr}\, \Phi(E_{ij}) = \delta_{ij}$.

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- (D. Drusvyatskiy, C.K. Li, D. Pelejo, Y.L. Voronin, H. Wolkowicz, 2015) General families. Construct a Choi matrix $P = (P_{ij}) \in M_n(M_m)$ such that

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- Evidently,

$$\{\text{Unitary operations}\} \subseteq \{\text{Mixed unitary operations}\}$$

$$\subseteq \{\text{Unital operations}\} \subseteq \{\text{General quantum operations}\}.$$

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Question For a given $\varepsilon > 0$, determine whether there is a quantum operation Φ such that $\|\Phi(\rho_j) - \sigma_j\| < \varepsilon$ for all $j = 1, \ldots, k$.

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  the trace norm $\|X\|_{tr} = {\rm tr}\,|X|$, and

  the Frobenius norm $\|X\|_{\rm Fr} = ({\rm tr}\,(X^*X))^{1/2}$.

- There are results on the upper bound and lower bounds for $d(\Phi(\rho_1), \sigma_1)$ for $\Phi \in \mathcal{S}$, where

  $\mathcal{S}$ is the set of all unitary, mixed unitary, unital, or general channels, and

  $d(\alpha, \beta)$ are different measures such as

    $\|\alpha - \beta\|$ for a unitary similarity invariant norm $\|\cdot\|$,

    the Fedility function $d(\alpha, \beta) = F(\alpha, \beta)$,

    the relative entropy function $d(\alpha, \beta) = S(\alpha||\beta)$.

- We first describe results on $\mathcal{F}_1 = \{A\}$ and $\mathcal{F}_2 = \{B\}$.

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$$a_1 \geq \cdots \geq a_n \quad \text{and} \quad b_1 \geq \cdots \geq b_n.$$

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- $\min \|\sigma_1 - \Phi(\rho_1)\|$ occurs if and only if there is a unitary U such that

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$$U\sigma_1 U^* = \mathrm{diag}\,(a_1, \ldots, a_n) \quad \text{and} \quad U\Phi(\rho_1)U^* = \mathrm{diag}\,(0, \ldots, 0, 1).$$

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2. There are unitary matrices $U_1, \ldots, U_n \in M_n$ such that

$$\sigma = \frac{1}{n}\left(U_1 \rho U_1^* + \cdots + U_n \rho U_n^*\right).$$

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Theorem [Li and Poon, 2011]

Let $\rho, \sigma \in M_n$ be density matrices. The following are equivalent.

1. There exists a mixed unitary quantum channel Φ such that $\Phi(\rho) = \sigma$.
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Theorem (based on a result in [Li & Tsing, 1989])

Let $\|\cdot\|$ be a USI norm, σ_1, ρ_1 are density matrices with eigenvalues

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Step 2. Let $2 \leq j < k \leq \ell \leq n$ be such that

$$\Delta_{j-1} \neq \Delta_j = \cdots = \Delta_{k-1} < \Delta_k = \cdots = \Delta_\ell \neq \Delta_{\ell+1}.$$

Replace each $\Delta_j, \ldots, \Delta_\ell$ by $(\Delta_j + \cdots + \Delta_\ell)/(\ell - j + 1)$, and go to Step 1.

Examples

Here are two examples illustrating the algorithm in the theorem.

Example 1 Let $\sigma_1 = \frac{1}{10}\text{diag}\,(4,3,3,0)$ and $\rho_1 = \frac{1}{10}\text{diag}\,(3,3,3,1)$.

Apply Step 0:

Set $(\Delta_1, \ldots, \Delta_4) = \frac{1}{10}\text{diag}\,(4,3,3,0) - \frac{1}{10}\text{diag}\,(3,3,3,1) = \frac{1}{10}\text{diag}\,(1,0,0,-1)$.

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Suppose ρ_1, ρ_2 have eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$.

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$$U\rho_1 U^* = \mathrm{diag}\,(a_1, \ldots, a_n), \qquad U\Phi(\rho_2)U^* = \mathrm{diag}\,(d_1, \ldots, d_n),$$

where $d_1, \ldots, d_n$ are determined as follows.

Step 0. Suppose $a_1 \geq \cdots \geq a_r \geq 0 = a_{r+1} = \cdots = a_n$. Let

$$a = (a_1, \ldots, a_r), \quad b = (b_1, \ldots, b_r), \quad (d_{r+1}, \ldots, d_n) = (b_{r+1}, \ldots, b_n).$$

Go to Step 1.

Step 1. Let $k \in \{1, \ldots, r\}$ be the largest positive integer such that

$$\frac{1}{a_1 + \cdots + a_k}(a_1, \ldots, a_k) \prec \frac{1}{b_1 + \cdots + b_k}(b_1, \ldots, b_k).$$

Set

$$(d_1, \ldots, d_k) = \frac{a_1 + \cdots + a_k}{b_1 + \cdots + b_k}(a_1, \ldots, a_k).$$

If $k = r$, then exit. Else, replace r, a, b by $r - k, (a_{k+1}, \ldots, a_r), (b_{k+1}, \ldots, b_r)$ and go to Step 1.

Examples If $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$, then $(d_1, \ldots, d_n) = (a_1, \ldots, a_n)$.

If $(b_1, \ldots, b_n) = (1/n, \ldots, 1/n)$, then $(d_1, \ldots, d_n) = (1/n, \ldots, 1/n)$.

More results and questions

- We also obtained results for general quantum channels, and other functions on two density matrices such as the relative entropy:

$$S(\rho_1||\rho_2) = \mathrm{tr}\, \rho_1(\log_2 \rho_1 - \log_2 \rho_2).$$

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- Minimize/maximize $d((\Phi(\rho_1), \ldots, \Phi(\rho_k)), (\sigma_1, \ldots, \sigma_k))$ for other distance measure d?
- one may start with the study of $\|\Phi(\rho_1 + i\rho_2) - (\sigma_1 + i\sigma_2)\|$ for the a special norm.

Thank you for your attention!