Riordan arrays and orthogonal polynomials

Combinatorial method in the analysis of algorithm and data structures, SKKU, Korea

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Motivating example 1

Permutations that avoid the *consecutive* pattern $\underline{123}$ are counted by a sequence that begins

$$1, 1, 2, 5, 17, 70, 349, 2017, 13358, 99377, 822041, \ldots$$

which has an exponential generating function given by

$$\frac{\frac{\sqrt{3}}{2} e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right)}.$$

These numbers are the moments associated to a family of orthogonal polynomials whose coefficient array is given by the exponential Riordan array

$$\left[\frac{\frac{\sqrt{3}}{2} e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right)}, \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right) - \frac{1}{2}\right]^{-1}.$$

The Riordan array

$$\left[\frac{\frac{\sqrt{3}}{2}e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right]$$

begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 5 & 12 & 6 & 1 & 0 & 0 & 0 \\ 17 & 53 & 39 & 10 & 1 & 0 & 0 \\ 70 & 279 & 260 & 95 & 15 & 1 & 0 \\ 349 & 1668 & 1914 & 880 & 195 & 21 & 1 \end{array}\right).$$

We have

$$\left[\frac{\frac{\sqrt{3}}{2}e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right]^{-1}$$

$$=\left[\frac{e^{\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)}}{\sqrt{1+x+x^2}}, \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)-\frac{\pi}{3\sqrt{3}}\right].$$

This begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & -6 & 1 & 0 & 0 & 0 \\ -13 & 4 & 21 & -10 & 1 & 0 & 0 \\ 49 & -129 & -5 & 55 & -15 & 1 & 0 \\ 31 & 723 & -624 & -85 & 120 & -21 & 1 \end{array}\right).$$

Motivating example 2

The production matrix of the exponential Riordan array

$$\left[\frac{\frac{\sqrt{3}}{2}e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right]$$

begins

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & 1 & 0 \\ 0 & 0 & 0 & 16 & 5 & 1 \\ 0 & 0 & 0 & 0 & 25 & 6 \end{array}\right).$$

The corresponding family of orthogonal polynomials satisfies the three-term recurrence

$$P_n(x) = P_n(x) = (x-n)P_{n-1}(x) - (n-1)^2 P_{n-2}(x),$$

with $P_0(x) = 1, P_1(x) = x - 1$.

Motivating example 3

The ordinary generating function of the moments $1, 1, 2, 5, 17, \ldots$ may be expressed as the continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{4x^2}{1-3x-\cfrac{9x^2}{1-4x-\cdots}}}}.$$

The Hankel transform of the moments $1, 1, 2, 5, 17, \ldots$ is given by

$$h_n = \prod_{k=0}^{n} (k+1)^{2(n-k)} = \prod_{k=0}^{n} k!^2.$$

The sequence

$$1, 1, 2, 5, 17, 70, 349, 2017, 13358, 99377, 822041, \ldots$$

is sequence A049774 in the **On-Line Encyclopedia of Integer Sequences**, created and maintained by Neil Sloane, available at oeis.org.

The sequence $h_n = 1, 1, 4, 144, 82944, \ldots$ is A055209.

Motivating example 4

The (n, k)-th element of the related exponential Riordan array

$$\left[\frac{3}{2\left(\cos\left(\sqrt{3}x+\frac{\pi}{3}\right)\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right]$$

counts k forests of planar increasing unary-binary trees with n nodes. Its inverse is the coefficient array of the family of orthogonal polynomials

$$P_n(x)=(x-n)P_{n-1}(x)-n(n-1)P_{n-2}(x),$$

with $P_0(x)=1, P_1(x)=x-1$.

Motivating example 5

If a_n is a given sequence, its binomial transform is the sequence

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

If $A(x)$ is the ordinary generating function of a_n, then $B(x)$ is given by

$$B(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) \cdot A(x).$$

If $A_e(x)$ is the exponential generating function of a_n, then

$$B_e(x) = e^x A(x) = [e^x, x]\,.A(x).$$

The Hankel transform of a_n and b_n are the same.

$\boxed{\textbf{Preliminaries on orthogonal polynomials}}$

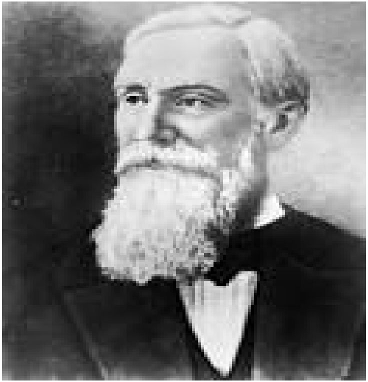

Figure: Pafnuty Chebyshev (1821 - 1894)

Orthogonal polynomials

Let $P_n(x)$ be a sequence of polynomials that obey a three-term recurrence

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x),$$

with $\beta_0 P_{-1}(x) = 0$ and $P_0(x) = 1$. Then $P_n(x)$ is a family of (monic) orthogonal polynomials. We have

$$\int P_n P_m d\,\mu(x) = \delta_{mn},$$

for an appropriate measure $\mu(x)$. Letting $a_n = \int x^n d\mu(x)$ then, for instance,

$$P_2(x) = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ 1 & x & x^2 \end{vmatrix} / \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} = h_2(x)/h_1$$

tau-function

We have

$$\beta_n = \frac{h_{n-1} h_{n+1}}{h_n^2}$$

and

$$\alpha_n = \frac{h^*_{n+1}}{h_{n+1}} - \frac{h^*_n}{h_n},$$

where for instance

$$h^*_2 = \begin{vmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \end{vmatrix}.$$

Figure: Thomas Joannes Stieltjes (1856-1894)

Moments and weight function

The generating function $g_\mu(x)$ of the moments μ_n can be obtained by

$$g_\mu(x) = \int_{-\infty}^{\infty} \frac{d\mu(z)}{1-xz}.$$

(Stieltjes or Cauchy transform).

In the reverse direction, we have the inversion formula

$$\mu((s,t)) + \frac{\mu(\{s\}) + \mu(\{t\})}{2} = \lim_{y \to +0} \int_s^t \operatorname{Im} G(x+iy)\, dx,$$

where

$$G(x) = \frac{1}{x} g_\mu\left(\frac{1}{x}\right).$$

(Stieltjes-Perron inversion formula). If $w(x)dx$ is the absolutely continuous part of μ, then

$$w(x) = -\frac{1}{\pi} \lim_{y \to +0} \operatorname{Im} G(x+iy).$$

Chebyshev polynomials of the second kind

We have

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}.$$

These orthogonal polynomials satisfy the three-term recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

with $U_0(x) = 1, U_1(x) = 2x$.
The measure for these polynomials is

$$d\mu(x) = \frac{1}{\pi}\sqrt{1-x^2}dx \qquad \text{on} \quad [-1, 1].$$

These polynomials begin

$$1, 2x, 4x^2 - 1, 8x^3 - 4x, 16x^4 - 12x^2 + 1, \ldots.$$

The normalized moments $\frac{2}{\pi}\int_{-1}^{1} x^n\sqrt{1-x^2}\,dx$ begin

$$1, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{5}{64}, 0, \frac{7}{128}, 0, \frac{21}{512}, 0, \ldots.$$

These are given by

$$\mu_n = \frac{n!}{2^n\left(\frac{n}{2}\right)!\left(\frac{n}{2}+1\right)!}\frac{1+(-1)^n}{2}.$$

For instance, we have

$$4\begin{vmatrix} 1 & 0 & 1/4 \\ 0 & 1/4 & 0 \\ 1 & x & x^2 \end{vmatrix} / \begin{vmatrix} 1 & 0 \\ 0 & 1/4 \end{vmatrix} = 4x^2-1.$$

Preliminaries on generating functions

Figure: Abraham deMoivre (1667-1754)

Generating functions

If a_n is the sequence

$$a_0, a_1, a_2, a_3, a_4 \ldots$$

then the expression

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is called the ordinary generating function of a_n.
The expression

$$A_e(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x}{2!} + \cdots$$

is called the exponential generating function of a_n.
We have

$$a_n = [x^n]A(x) = n![x^n]A_e(x),$$

where $[x^n]$ is the operator that extracts the coefficient of x^n from a generating function.

General generating function

If c_n is a sequence such that $c_n \neq 0$ for all n, then we can define

$$A_c(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{c_n} = \frac{a_0}{c_0} + a_1 \frac{x}{c_1} + a_2 \frac{x}{c_2} + \cdots .$$

We then have

$$a_n = c_n [x^n] A_c(x).$$

For the ordinary generating function, we have $c_n = 1$ for all n.
For the exponential generating function, we have $c_n = n!$.
Other choices might be $c_n = (n+1)!$, or $c_n = 2^n n!$.
Mathematical physics is a good source of alternative values for c_n.

Let $\phi(t) = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!}$. Then

$$\begin{aligned}
\int_0^{\infty} \phi(t) e^{-tx}\, dt &= \int_0^{\infty} \left(\sum_{k=0}^{\infty} c_k \frac{t^k}{k!} \right) e^{-tx}\, dt \\
&= \sum_{k=0}^{\infty} c_k \int_0^{\infty} \frac{t^k e^{-tx}}{k!}\, dt \\
&= \sum_{k=0}^{\infty} c_k x^{-(k+1)} \\
&= \frac{1}{x} g\left(\frac{1}{x}\right),
\end{aligned}$$

where

$$g(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Sumudu transform

To go from an exponential generating function to an ordinary generating function, we can use this variant of the Laplace transform.

$$g(x) = \frac{1}{x}\int_0^{\infty} \phi(t) e^{-\frac{t}{x}}\, dt$$

$\boxed{\text{Generating functions are useful when they can be written in a compact form}}$

Example

The ordinary generating function of the sequence

$$1, 1, 1, 1, 1, \ldots$$

is given by

$$\sum_{n=0}^{\infty} 1.x^n = \sum_{n=0}^{\infty} x^n.$$

A short way to write $\sum_{n=0}^{\infty} x^n$ is

$$\frac{1}{1-x}$$

This can be seen by

- Carrying out the long division of 1 by $1-x$
- Using the extended binomial theorem

Convolutions

If

$$f(x)=\sum_{n=0}^{\infty} f_n x^n$$

and

$$g(x)=\sum_{n=0}^{\infty} g_n x^n,$$

then

$$f(x)g(x)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} f_k g_{n-k}\right)x^n.$$

If

$$f(x)=\sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

and

$$g(x)=\sum_{n=0}^{\infty} g_n \frac{x^n}{n!},$$

then

$$f(x)g(x)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}\right)\frac{x^n}{n!}.$$

(Please correct notes!)

The Binomial Theorem

Binomial theorem

We have the well known formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

$$(a+b)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k b^{n-k}.$$

Now we use

$$\binom{-n}{k} = (-1)^n \binom{n+k-1}{k}.$$

$$\begin{aligned}
[x^n]\frac{1}{1-x} &= [x^n](1-x)^{-1} \\
&= [x^n]\sum_{k=0}^{\infty}\binom{-1}{k}(-1)^k x^k.1^{n-k} \\
&= [x^n]\sum_{k=0}^{\infty}\binom{k+1-1}{k}x^k \\
&= [x^n]\sum_{k=0}^{\infty}\binom{k}{k}x^k = 1.
\end{aligned}$$

Binomial example

$$\begin{aligned}
[x^n]\frac{x^k}{(1-x)^{k+1}} &= [x^{n-k}](1-x)^{-(k+1)} \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{-(k+1)}{j}(-x)^j \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}x^j \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j}{j}x^j \\
&= \binom{k+n-k}{n-k} \\
&= \binom{n}{n-k}=\binom{n}{k}.
\end{aligned}$$

Lagrange Inversion

Series reversion

Let

$$f(x) = 0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots .$$

The solution to

$$f(u) = x$$

with $u(0) = 0$ is called the reversion of f. We shall denote it by

$$\bar{f}(x) = \mathrm{Rev}\{f\}(x).$$

We have

$$\bar{f}(f(x)) = x \quad \text{and} \quad f(\bar{f}(x)) = x.$$

We also have

$$\mathrm{Rev}\{\mathrm{Rev}\{f\}\}(x) = f(x) \quad \text{or} \quad \bar{\bar{f}}(x) = x.$$

$\boxed{\text{\textbf{Lagrange inversion} allows us to extract the coefficients of } \bar{f}(x) \text{ using a knowledge of those of } f.}$

Lagrange inversion

We have

$$[x^n]G(\bar{f}(x)) = \frac{1}{n}[x^{n-1}]G'(x)\left(\frac{x}{f(x)}\right)^n.$$

Equivalently, we have

$$[x^n]G(f(x)) = \frac{1}{n}[x^{n-1}]G'(x)\left(\frac{x}{\bar{f}(x)}\right)^n.$$

Lagrange inversion

Let $f(x) = x(1-x)$. Then

$$\bar{f}(x) = \frac{1-\sqrt{1-4x}}{2}.$$

$$\begin{aligned}
[x^n](\bar{f}(x))^k &= \frac{1}{n}[x^{n-1}]kx^{k-1}\left(\frac{x}{x(1-x)}\right)^n \\
&= \frac{k}{n}[x^{n-k}]\left(\frac{1}{1-x}\right)^n \\
&= \frac{k}{n}[x^{n-k}]\sum_{j=0}^{\infty}\binom{n+j-1}{j}x^j \\
&= \frac{k}{n}\binom{n+n-k-1}{n-k} \\
&= \frac{k}{n}\binom{2n-k-1}{n-k}.
\end{aligned}$$

$\boxed{\textbf{Continued Fractions}}$

Continued fractions

Sometimes, generating functions can be written as S-continued fractions. Consider

$$f(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cdots}}}}.$$

If we let $u = f(x)$, then we have

$$u = \frac{1}{1 - xu}.$$

Thus

$$u(1 - xu) = 1 \quad \text{or} \quad xu^2 - u - 1 = 0.$$

We obtain

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

The following type of J-continued fraction

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}$$

can be associated to lattice paths.

Equivalent forms

$$\cfrac{a_0}{1-\cfrac{a_1 x}{1-\cfrac{a_2 x}{1-\cfrac{a_3 x}{1-\cdots}}}}$$

is equal to

$$\cfrac{a_0}{1-a_1 x-\cfrac{a_1 a_2 x^2}{1-(a_2+a_3)x-\cfrac{a_3 a_4 x^2}{1-(a_4+a_5)x-\cdots}}}.$$

Other forms are common. For instance,

$$\cfrac{1}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-x-\cdots}}}}$$

is equal to

$$S(x)=\frac{1-x-\sqrt{1-6x+x^2}}{2x},$$

which expands to give

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots,$$

the large Schroeder numbers.

In this case, we also have

$$S(x) = \cfrac{1}{1 - 2x - \cfrac{2x^2}{1 - 3x - \cfrac{2x^2}{1 - 3x - \cfrac{2x^2}{1 - 3x - \cdots}}}}.$$

"These count the number of (colored) Motzkin n-paths with each up-step and each flat-step at ground level getting one of 2 colors and each flat-step not at ground level getting one of 3 colors".

Recurrences

Recurrences

Sequences can often be described by recurrences, where we prescribe how to construct the elements of the sequence in terms of known prior values.

Fibonacci numbers

The Fibonacci numbers F_n are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for} \quad n \geq 2,$$

with $F_0 = 0$, $F_1 = 1$.
Thus each Fibonacci number is the sum of the two proceeding numbers, beginning with $0, 1$. We get

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

Multiplying by x^n and summing from $n = 2$ on, we get

$$\begin{aligned} \sum_{n=2}^{\infty} F_n x^n &= \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n \\ &= x \sum_{n=0}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \end{aligned}$$

Adding $F_0 x^0 + F_1 x^1 = 0 \cdot x^0 + 1 \cdot x = 0 + x$ to both sides and simplifying, we get

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}.$$

Catalan numbers

The Catalan numbers C_n which begin

$$1, 1, 2, 5, 14, 42, \ldots,$$

satisfy the convolution-type recurrence

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad n \geq 1,$$

with $C_0 = 1$.
From this we can show that

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

The Stirling numbers of the second kind

The Stirling numbers of the Second kind, $S(n,k) = \left\{ {n \atop k} \right\}$, satisfy the recurrence

$$S(n,k) = S(n,k-1) + kS(n,k),$$

with the initial conditions

$$S(0,0) = 1, \quad \text{and} \quad S(n,0) = S(0,n) = 0, n > 0.$$

The Stirling numbers of the second kind $S(n,k)$ count forests of k increasing unary trees on n nodes (as well as the number of ways to partition a set of n objects into k non-empty subsets).

An interesting recurrence

$$a_{n+2}a_n = a_{n+1}^2 + 1,$$

or more generally,

$$a_n = \frac{a_{n-1}^2 + s}{a_{n-2}}.$$

Remark: With $a_0 = 1, a_1 = r$, and $s = rk + r - 1$, the solutions are integer valued and are linked to special Riordan arrays.

Polynomial families

Polynomial families - 1

By a polynomial family $P_n(x)$ we shall understand a sequence of polynomials

$$P_0(x), P_1(x), P_2(x), P_3(x), \ldots$$

where $P_n(x)$ is of exact degree n. We thus have

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

The infinite matrix $(a_{n,k})_{n,k \geq 0}$ will then be a lower-triangular matrix. We call this matrix the *coefficient array* of the family of polynomials.

Polynomial families - 2

Consider the polynomial family given by

$$P_n(x) = (1+x)^n.$$

By the binomial theorem, we have

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Thus the coefficient array in this case is the binomial matrix (Pascal's triangle)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Thus we have

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}\right)\left(\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{array}\right)=\left(\begin{array}{c} 1 \\ 1+x \\ (1+x)^2 \\ (1+x)^3 \\ (1+x)^4 \\ (1+x)^5 \end{array}\right)$$

Polynomial families - 3

Consider the family of polynomials

$$P_n(x) = \prod_{k=0}^{n-1} (x+k).$$

The family begins

$$1, x, x(x+1), x(x+1)(x+2), \ldots$$

We have

$$P_n(x) = \sum_{k=0}^{n} S(n,k) x^k$$

where $S(n,k) = \left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind.

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 6 & 11 & 6 & 1 \end{array} \right).$$

Thus we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 6 & 11 & 6 & 1 & 0 \\ 0 & 24 & 50 & 35 & 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x(x+1) \\ x(x+1)(x+2) \\ x(x+1)(x+2)(x+3) \\ x(x+1)(x+2)(x+3)(x+4) \end{pmatrix}$$

Orthogonal polynomials and continued fractions

A generating function of the form

$$g(x) = \sum_{k=0}^{\infty} \mu_n x^n$$

where

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}$$

can be associated to the family of polynomials that obey

$$P_n(x) = (x - \alpha_n) P_{n-1}(x) - \beta_n P_{n-1}(x).$$

Hankel transform

Hankel transform

Given a sequence μ_n, we can define its Hankel transform to be the sequence h_n where

$$h_n = |\mu_{i+j}|_{0 \le i,j \le n}.$$

If μ_n has a generating function

$$\cfrac{\mu_0}{1-\alpha_0 x - \cfrac{\beta_1 x^2}{1-\alpha_1 x - \cfrac{\beta_2 x^2}{1-\alpha_2 x - \cfrac{\beta_3 x^2}{1-\alpha_3 x - \cdots}}}}$$

then we have

$$h_n = \mu_0^{n+1} \prod_{k=0}^{n} \beta_k^{n-k}.$$

Note that this is independent of the α_n.

Figure: Lou Shapiro

Discrete Applied Mathematics 34 (1991) 229–239
North-Holland

The Riordan group

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Abstract

Shapiro, L.W., S. Getu, W.-J. Woan and L.C. Woodson, The Riordan group, Discrete Applied Mathematics 34 (1991) 229–239.

Introduction

The central concept in this article is a group which we call the Riordan group. With the recent death of John Riordan this seems appropriate to name after him.

Preliminaries on Riordan arrays

Riordan arrays - ordinary

Let

$$g(x) = g_0 + g_1 x + g_2 x^2 + \ldots = \sum_{n=0}^{\infty} g_n x^n$$

$$f(x) = 0 + f_1 x + f_2 x^2 + \ldots = \sum_{n=1}^{\infty} f_n x^n.$$

The matrix with (n, k)-th element given by

$$t_{n,k} = [x^n] g(x) f(x)^k$$

is called *the (ordinary) Riordan array* $(g(x), f(x))$ defined by the pair $g(x), f(x)$.

Riordan arrays - exponential

Let

$$g(x) = g_0 + g_1\frac{x}{1!} + g_2\frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}$$

$$f(x) = 0 + f_1\frac{x}{1!} + f_2\frac{x^2}{2!} + \ldots = \sum_{n=1}^{\infty} f_n \frac{x^n}{n!}.$$

The matrix with (n, k)-th element given by

$$t_{n,k} = \frac{n!}{k!}[x^n]g(x)f(x)^k$$

is called *the exponential Riordan array* $[g(x), f(x)]$ defined by the pair $g(x), f(x)$.

Riordan arrays and combinatorial structures

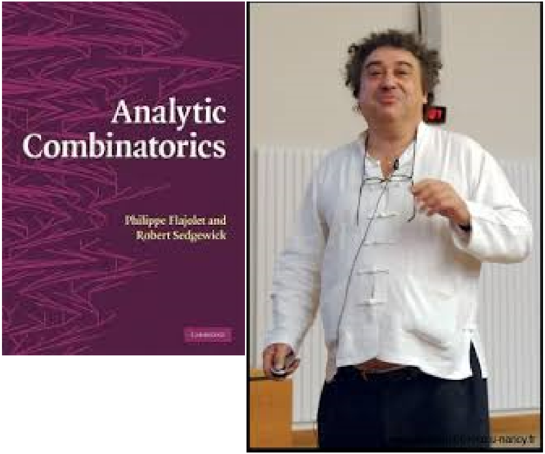
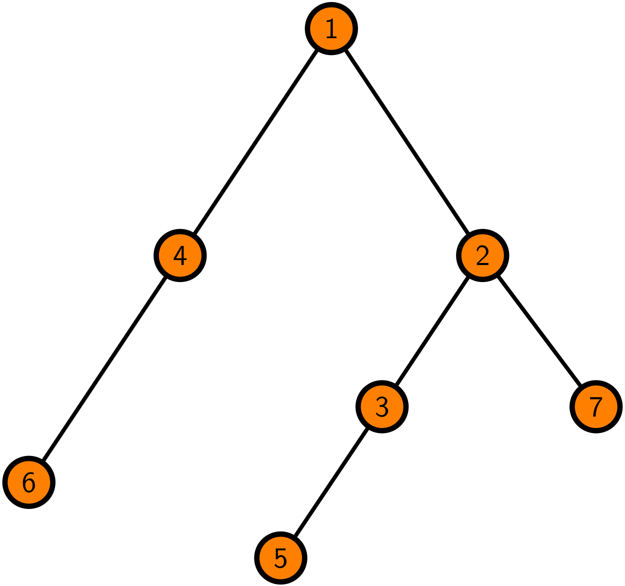


Figure: Phillipe Flajolet (1948 - 2011)

Increasing trees

"An increasing tree is a labelled rooted tree in which labels along any branch from the root go in increasing order. Such trees can represent permutations, data structures in computer science, and probabilistic models in diverse applications." (Bergeron, Flajolet, Salvy)


1
4
2
3
7
6
5

Degree weight generating function $\phi(x)$

For non-planar graphs, we have

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n \frac{x^n}{n!},$$

where there are ϕ_n sorts of nodes of outdegree n.
We can associate the following expressions with increasing trees of different kinds.

- plane binary: $\phi(x) = (1+x)^2$
- Motzkin (plane unary-binary): $\phi(x) = 1 + x + x^2$
- non plane unary-binary: $\phi(x) = 1 + x + x^2/2$
- general Catalan tree: $\phi(x) = \frac{1}{1-x}$
- non plane recursive (Cayley): $\phi(x) = e^x$

Production matrix

For each $\phi(x)$, we can consider the matrix with bivariate generating function

$$e^{xy}(\phi'(x) + y\phi(x)).$$

We take the example of

$$\phi(x) = 1 + x + x^2 \Longrightarrow \phi'(x) = 1 + 2x.$$

Thus we consider the matrix with bivariate generating function

$$e^{xy}(1 + 2x + y(1 + x + x^2)).$$

We understand this to be exponential in x, and ordinary in y.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t_{n,k}\frac{x^n}{n!}y^k = e^{xy}(1 + 2x + y(1 + x + x^2)).$$

Jacobi matrix

In the case of $\phi(x) = 1 + x + x^2$, we obtain

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 20 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 30 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 42 & 7 \end{pmatrix}.$$

Jacobi matrix

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 20 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 30 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 42 & 7 \end{array}\right).$$

Dividing each element $t_{n,k}$ by $\frac{n!}{k!}$, we obtain the matrix

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 7 & 7 \end{array}\right).$$

Thus in this case, each diagonal is in arithmetic progression.

We have

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & 12 & 4 \end{pmatrix}^0 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & 12 & 4 \end{pmatrix}^1 = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & 12 & 4 \end{pmatrix}.$$

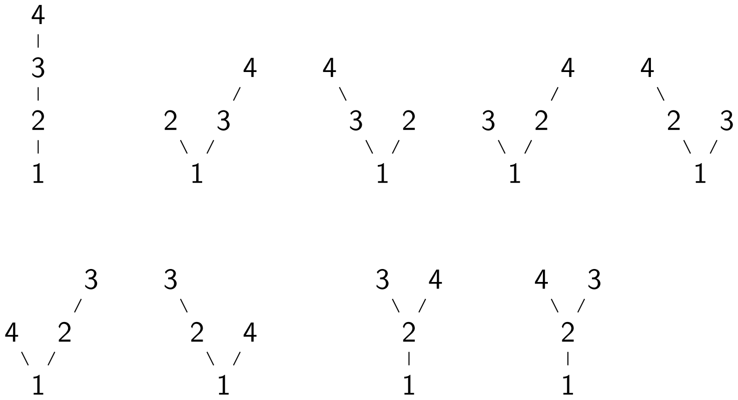
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & 12 & 4 \end{pmatrix}^2 = \begin{pmatrix} \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 \\ 6 & 12 & 5 & 1 \\ 12 & 30 & 27 & 7 \\ 0 & 72 & 84 & 28 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & 12 & 4 \end{pmatrix}^3 = \begin{pmatrix} \mathbf{9} & \mathbf{15} & \mathbf{6} & \mathbf{1} \\ 30 & 60 & 39 & 9 \\ 72 & 234 & 195 & 55 \\ 144 & 648 & 660 & 196 \end{pmatrix}.$$

The production matrix generates the following lower-triangular matrix.

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 9 & 15 & 6 & 1 & 0 & 0 & 0 \\ 39 & 75 & 45 & 10 & 1 & 0 & 0 \\ 189 & 459 & 330 & 105 & 15 & 1 & 0 \\ 1107 & 3087 & 2709 & 1050 & 210 & 21 & 1 \end{array}\right).$$

The sequence $1, 1, 3, 9, 39, 189, 1107, \ldots$ is A080635(n+1), the number of permutations on $n+1$ letters without double falls and without initial falls.



The sequence $1, 1, 1, 3, \mathbf{9}, 39, 189, 1107, \ldots$ counts the number of planar increasing unary-binary trees with n nodes.

We have

$$\int_0^x \frac{1}{\phi(t)}\,dt = \int_0^x \frac{1}{1+t+t^2}\,dt = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}}.$$

Now solving the equation

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2z}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}} = x$$

for z, we find that

$$z = \operatorname{Rev}\left(\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}}\right) = \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right) - \frac{1}{2}.$$

Furthermore, we have

$$\int_0^x \frac{\phi'(t)}{\phi(t)}\,dt = \int_0^x \frac{1+2t}{1+t+t^2}\,dt = \ln(1+x+x^2).$$

We let

$$g(x) = e^{-\ln(1+x+x^2)} = \frac{1}{1+x+x^2}.$$

Now form

$$\frac{1}{g\left(\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right)}$$

to get

$$\frac{3}{2(\cos\left(\sqrt{3}x+\frac{\pi}{3}\right)+1)}.$$

This is the e.g.f. of the sequence $1, 1, 3, 9, 39, 189, \ldots$.

The (ordinary) generating function of the sequence $1, 1, 3, 9, 39, 189, \ldots$ can be expressed as the continued fraction

$$\cfrac{1}{1-x-\cfrac{2x^2}{1-2x-\cfrac{6x^2}{1-3x-\cfrac{12x^2}{1-4x-\cdots}}}}.$$

Thus the Hankel transform of this sequence is given by

$$h_n = \prod_{k=0}^{n} ((k+1)(k+2))^{n-k}.$$

The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 9 & 15 & 6 & 1 & 0 & 0 & 0 \\ 39 & 75 & 45 & 10 & 1 & 0 & 0 \\ 189 & 459 & 330 & 105 & 15 & 1 & 0 \\ 1107 & 3087 & 2709 & 1050 & 210 & 21 & 1 \end{pmatrix}$$

is the exponential Riordan array

$$\left[\frac{3}{2\left(\cos\left(\sqrt{3}x+\frac{\pi}{3}\right)+1\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right]$$

or

$$\left[\frac{d}{dx}\text{Rev}\int_0^x \frac{1}{\phi(t)}\,dt, \text{Rev}\int_0^x \frac{1}{\phi(t)}\,dt\right].$$

The sequence $1, 1, 3, 9, 39, 189, 1107, \ldots$ is the moment sequence for the family of orthogonal polynomials

$$P_n(x) = (x-n)P_{n-1}(x) - n(n-1)P_{n-2}(x),$$

with $P_0(x) = 1$, $P_1(x) = x - 1$.
The coefficient array of this family of polynomials is given by

$$\left[\frac{3}{2\left(\cos\left(\sqrt{3}x + \frac{\pi}{3}\right) + 1\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right) - \frac{1}{2}\right]^{-1}$$

$$= \left[\frac{1}{1+x+x^2}, \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}}\right].$$

We now take $\phi(x) = 1 + x^2$. Then $\phi'(x) = 2x$.
The expression

$$e^{xy}(2x + y(1 + x^2))$$

expands to give the production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 42 & 0 \end{pmatrix}$$

corresponding to

$$P_n(x) = xP_{n-1}(x) - n(n-1)P_{n-2}(x).$$

The production matrix generates the matrix

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 1 & 0 & 0 & 0 \\ 16 & 0 & 20 & 0 & 1 & 0 & 0 \\ 0 & 136 & 0 & 40 & 0 & 1 & 0 \\ 272 & 0 & 616 & 0 & 70 & 0 & 1 \end{array}\right),$$

which is

$$\left[\frac{1}{\cos^2(x)}, \tan(x)\right] = \left[\frac{1}{1+x^2}, \tan^{-1}(x)\right]^{-1}.$$

The numbers $1, 2, 16, 272, 7936, 353792, 22368256, \ldots$ are the tangent numbers (or "Zag" numbers).

Non plane unary binary trees

For $\phi(x) = 1 + x + x^2/2$ (non-plane unary binary trees), we have

$$\int_0^x \frac{dt}{\phi(t)} = 2\tan^{-1}(1+x) - \frac{\pi}{2},$$

and

$$\int_0^x \frac{\phi'(t)dt}{\phi(t)} = \ln\left(\frac{2+2x+x^2}{2}\right).$$

We find that

$$\left[\frac{2}{2+2x+x^2}, 2\tan^{-1}(1+x) - \frac{\pi}{2}\right]^{-1} = \left[\frac{1}{1-\sin(x)}, \tan\left(\frac{2x+\pi}{4}\right) - 1\right]$$

is the coefficient array of the corresponding family of orthogonal polynomials. These are

$$P_n(x) = (x-n)P_{n-1}(x) - \binom{n}{2}P_{n-2}(x).$$

Non plane unary binary trees

The production matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 15 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 21 & 7 \end{pmatrix}$$

has generating function

$$e^{xy}(1+x+y(1+x+x^2/2)).$$

Non plane unary binary trees

The moment matrix

$$\left[\frac{1}{1-\sin(x)}, \tan\left(\frac{2x+\pi}{4}\right)-1\right]$$

begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 5 & 11 & 6 & 1 & 0 & 0 & 0 \\ 16 & 45 & 35 & 10 & 1 & 0 & 0 \\ 61 & 211 & 210 & 85 & 15 & 1 & 0 \\ 272 & 1113 & 1351 & 700 & 175 & 21 & 1 \end{pmatrix}.$$

It enumerates forests of k increasing unordered trees on the vertex set $\{1, 2, \ldots, n\}$ rooted at 1, in which all outdegrees are ≤ 2.

- Increasing trees $\Longrightarrow$ Exponential Riordan arrays $[g, f]$ where $f'(x) = g(x)$.
- When $\phi(x) = a + bx + cx^2$ the inverses of these exponential Riordan arrays are the coefficient arrays of orthogonal polynomials.
- When $\phi(x)$ is a polynomial of degree d, we have $(d-1)$-orthogonal polynomials

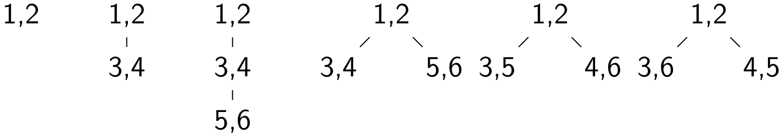
Bilabelled increasing trees

The exponential Riordan array

$$\left[\frac{1}{\cos^2(x/\sqrt{2})}, \sqrt{2}\tan\left(\frac{x}{\sqrt{2}}\right)\right] = \left[\frac{2}{2+x^2}, \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]^{-1}$$

is associated to unordered bilabelled increasing trees.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 \\ 4 & 0 & 10 & 0 & 1 \end{pmatrix}$$



Production matrix; orthogonal polynomials

$$\left(\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 28 & 0 \end{array}\right)$$

$$e^{xy}\left(x+y\left(1+\frac{x^2}{2}\right)\right)$$

$$P_n(x) = xP_{n-1}(x) - \frac{n(n-1)}{2}P_{n-2}(x),$$

$P_0(x) = 1,\ P_1(x) = x.$

Ordered bilabelled increasing trees

The first column of the exponential Riordan array

$$\left[e^{\mathrm{InvErf}^2\left(\sqrt{\frac{2}{\pi}}x\right)}, \sqrt{2}\,\mathrm{InvErf}\left(\sqrt{\frac{2}{\pi}}x\right)\right]$$

counts ordered bilabelled increasing trees

$$1, 1, 7, 127, 4369, \ldots.$$

Its production matrix has g.f. given by

$$e^{xy}(xe^{x^2/2} + ye^{x^2/2}).$$

The production matrix in this case is the "beheaded" exponential Riordan array

$$\left[e^{x^2/2}, x\right].$$

Back to 123

The matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & 1 & 0 \\ 0 & 0 & 0 & 16 & 5 & 1 \\ 0 & 0 & 0 & 0 & 25 & 6 \end{pmatrix}$$

is generated by

$$e^{xy}(1+x+y(1+x+x^2))$$

Riordan array theory allows us to go from the pair

$$(1+x, 1+x+x^2)$$

to the exponential Riordan array

$$\left[\frac{\frac{\sqrt{3}}{2}e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right].$$

We know that

$$\int_0^x \frac{dt}{1+t+t^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}},$$

and that

$$\text{Rev}\left\{\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}}\right\} = \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}x}{2} + \frac{\pi}{6}\right) - \frac{1}{2}.$$

We also have

$$\int_0^x \frac{1+t}{1+t+t^2}\, dt = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) + \frac{1}{2}\ln(1+x+x^2) - \frac{\pi}{6\sqrt{3}}.$$

Then

$$e^{-\int_0^x \frac{1+t}{1+t+t^2}\, dt} = \frac{e^{\frac{\pi}{6\sqrt{3}} - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)}}{\sqrt{1+x+x^2}}.$$

The matrix we seek is then

$$\left[\frac{e^{\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)}}{\sqrt{1+x+x^2}}, \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)-\frac{\pi}{3\sqrt{3}}\right]^{-1}$$

$$=\left[\frac{\frac{\sqrt{3}}{2}e^{x/2}}{\cos\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)}, \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)-\frac{1}{2}\right].$$

$$(1+x, 1+x+x^2) \Longrightarrow \left[\frac{e^{\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)}}{\sqrt{1+x+x^2}}, \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)-\frac{\pi}{3\sqrt{3}}\right]^{-1}$$

Permutations that avoid $\underline{123}$.

$$(1+2x, 1+x+x^2) \Longrightarrow \left[\frac{1}{1+x+x^2}, \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right)-\frac{\pi}{3\sqrt{3}}\right]^{-1}$$

Planar increasing unary-binary trees.

The pair $(1+3x, 1+x+x^2)$ lead to the sequence

$$1, 1, 4, 13, 67, 358, 2365, 17053, 139780, 1251865, 12318247, \ldots.$$

Any ideas what this counts?

The Riordan Group

Ordinary Riordan arrays

Given two power series

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots = \sum_{n=0}^{\infty} g_n x^n,$$

and

$$f(x) = 0 + f_1 x + f_2 x^2 + \cdots = \sum_{n=1}^{\infty} f_n x^n,$$

we define the associated Riordan array (g, f) to be the lower-triangular matrix with (n, k)-th term

$$t_{n,k} = [x^n] g(x) f(x)^k$$

Thus $t_{n,k}$ is the coefficient of x^n in the expansion of the product $g(x)f(x)^k$.

Method of coefficients

The coefficient extraction operator $[x^n]$ acts according to a number of simple rules.

Linearity	$[x^n](rf(x)+sg(x))$	$=$	$r[x^n]f(x)+s[x^n]g(x)$
Shifting	$[x^n]xf(x)$	$=$	$[x^{n-1}]f(x)$
Differentiation	$[x^n]f'(x)$	$=$	$(n+1)[x^{n+1}]f(x)$
Convolution	$[x^n]g(x)f(x)$	$=$	$\sum_{k=0}^n([x^k]g(x))[x^{n-k}]f(x)$
Composition	$[x^n]g(f(x))$	$=$	$\sum_{k=0}^{\infty}([x]^k g(x))[x^n]f(x)^k$
Inversion	$[x^n]\bar{f}(x)^k$	$=$	$\frac{k}{n}[x^{n-k}]\left(\frac{x}{f(x)}\right)^n$

Composition of power series

If

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots = \sum_{n=0}^{\infty} g_n x^n,$$

and

$$f(x) = 0 + f_1 x + f_2 x^2 + \cdots = \sum_{n=1}^{\infty} f_n x^n,$$

then the composition of g and f is defined by

$$g(f(x)) = g_0 + g_1 f(x) + g_2 f(x)^2 + g_3 f(x)^3 + \cdots = \sum_{n=0}^{\infty} g_n f(x)^k.$$

Series reversion

For a power series

$$f(x) = 0 + f_1 x + f_2 x + \cdots = \sum_{n=1}^{\infty} f_n x^n,$$

the reversion of f,

$$\bar{f} = \text{Rev} f,$$

is defined to be the power series such that

$$\bar{f}(f(x)) = f(\bar{f}(x)) = x.$$

Thus $\bar{f}(x)$ is the solution of the equation

$$f(u) = x$$

such that $u(0) = 0$.

Series reversion example

Example

We have seen that the reversion of $x(1-x)$ is obtained by solving

$$u(1-u)=x, \quad \text{or} \quad u^2-u+x=0.$$

We get

$$\overline{x(1-x)}=\text{Rev}\{x(1-x)\}=\frac{1-\sqrt{1-4x}}{2}.$$

Example

In like manner, the reversion of $\frac{x}{1-x}$ is obtained by solving

$$\frac{u}{1-u}=x \quad \text{or} \quad u=x-xu \quad \text{or} \quad u(1+x)=x.$$

$$\text{We get} \quad \overline{\frac{x}{1-x}}=\text{Rev}\{\frac{x}{1-x}\}=\frac{x}{1+x}.$$

Let us calculate $[x^n]\overline{x(1-x)}$. We have

$$\begin{aligned}
[x^n]\overline{x(1-x)} &= \frac{1}{n}[x^{n-1}]\left(\frac{x}{x(1-x)}\right)^n \\
&= \frac{1}{n}[x^{n-1}](1-x)^{-n} \\
&= \frac{1}{n}[x^{n-1}]\sum_{j=0}^{\infty}\binom{-n}{j}(-1)^j x^j \\
&= \frac{1}{n}[x^{n-1}]\sum_{j=0}^{\infty}\binom{n+j-1}{j}x^j \\
&= \frac{1}{n}\binom{n+n-1-1}{n-1} \\
&= \frac{1}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

Thus $\quad [x^n]\dfrac{1-\sqrt{1-4x}}{2x} = \dfrac{1}{n+1}\dbinom{2n}{n} = C_n.$

The Riordan group

We have $[x^n]x^k = \delta_{n,k}$ and so the Riordan array $(1, x)$ is the Identity matrix.
The set

$$\mathcal{R} = \{(g, f) \,|\, g = g_0 + g_1 x + \cdots, f = 0 + f_1 x + f_2 x^2 + \cdots\},$$

along with matrix multiplication, is then a group. In terms of the defining power series, matrix multiplication corresponds to the following rule:

$$(g(x), f(x)) \cdot (u(x), v(x)) = (g(x)u(f(x)), v(f(x))).$$

Inverse of a Riordan array

The inverse of $(g(x), f(x))$ is given by

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right).$$

The Riordan array $A = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ is the Binomial matrix (Pascal's triange) with general element $\binom{n}{k}$. The Riordan array $B = (1+x, x(1+x))$ is the matrix with general term $\binom{k+1}{n-k}$. Let us calculate AB and BA. We have

$$\begin{aligned}
\left(\frac{1}{1-x}, \frac{x}{1-x}\right) \cdot (1+x, x(1+x)) &= \left(\frac{1}{1-x}\left(1+\frac{x}{1-x}\right), \frac{x}{1-x}\left(1+\frac{x}{1-x}\right)\right) \\
&= \left(\frac{1}{1-x}\left(\frac{1}{1-x}\right), \frac{x}{1-x}\left(\frac{1}{1-x}\right)\right) \\
&= \left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right).
\end{aligned}$$

$$\begin{aligned}
&(1+x, x(1+x)) \cdot \left(\frac{1}{1-x}, \frac{x}{1-x}\right) \\
&\qquad = \left((1+x)\frac{1}{1-x(1+x)}, \frac{x(1+x)}{1-x(1+x)}\right) \\
&\qquad = \left(\frac{1+x}{1-x-x^2}, \frac{x(1+x)}{1-x-x^2}\right).
\end{aligned}$$

Some subgroups

The last two Riordan arrays are elements of the **Bell** subgroup of $\mathcal{R}$.

$$\mathrm{B} = \{(g, f) \in \mathcal{R} \,|\, f(x) = xg(x)\}.$$

or

$$\mathrm{B} = \{(g, f) \in \mathcal{R} \,|\, g(x) = f(x)/x\}.$$

Another subgroup is the **Derivative** subgroup

$$\mathcal{D} = \{(g, f) \in \mathcal{R} \,|\, g(x) = f'(x)\}.$$

The **Appell** subgroup is the group

$$\mathcal{A} = \{(g, f) \in \mathcal{R} \,|\, f(x) = x\}.$$

The **Lagrange** (or **associated**) subgroup is the group

$$\mathcal{L} = \{(g, f) \in \mathcal{R} \,|\, g(x) = 1\}.$$

The Appell subgroup

$$\mathcal{A} = \{(g, x) \,|\, g(x) = g_0 + g_1 x + g_2 x^2 + \cdots\}.$$

We have

$$t_{n,k} = [x^n]g(x)x^k = [x^{n-k}]g(x) = g_{n-k}$$

$$\left(\begin{array}{ccccc} g_0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0 & 0 & 0 & \cdots \\ g_2 & g_1 & g_0 & 0 & \cdots \\ g_3 & g_2 & g_1 & g_0 \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

We have

$$(g(x), x) \cdot (1, f(x)) = (g(x), f(x)).$$

Then we have the semi-direct product

$$\boxed{\mathcal{R} = \mathcal{A} \rtimes \mathfrak{L}}$$

where $\mathcal{A}$ is a *normal* subgroup of $\mathcal{R}$.

Fundamental Theorem of Riordan Arrays

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence a_n.
Let $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function of the sequence b_n, where we have

$$(g, f) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}.$$

Then

$$\begin{aligned} B(x) &= (g(x), f(x))A(x) \\ &= g(x)A(f(x)). \end{aligned}$$

Row sums

$$(g(x), f(x)) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} t_{0,0} & 0 & 0 & 0 & \cdots \\ t_{1,0} & t_{1,1} & 0 & 0 & \cdots \\ t_{2,0} & t_{2,1} & t_{2,2} & 0 & \cdots \\ t_{3,0} & t_{3,1} & t_{3,2} & t_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} t_{0,0} \\ t_{1,0} + t_{1,1} \\ t_{2,0} + t_{2,1} + t_{2,2} \\ \vdots \end{pmatrix}.$$

$$(g(x), f(x)) \frac{1}{1-x} = g(x) \frac{1}{1-f(x)} = \frac{g(x)}{1-f(x)}.$$

Binomial transform 1

Consider the Riordan array

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

We have

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right) = \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) = \left(\binom{n}{k}\right).$$

Binomial transform 2

$$\begin{aligned}
[x^n]\frac{1}{1-x}\left(\frac{x}{1-x}\right)^k &= [x^n]\frac{x^k}{(1-x)^{k+1}} \\
&= [x^{n-k}](1-x)^{-(k+1)} \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{-(k+1)}{j}(-1)^j x^j \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}x^j \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j}{j}x^j \\
&= \binom{k+n-k}{n-k} = \binom{n}{n-k} = \binom{n}{k}
\end{aligned}$$

Binomial transform 3

Let the sequence a_n have generating function $A(x)$. The binomial transform of a_n is the sequence b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

But

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right) \cdot A(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right).$$

Thus the binomial transform b_n of a_n will have generating $B(x)$ given by

$$B(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right).$$

A lattice path example

A *Dyck path* is a path in the first quadrant which begins at the origin $(0, 0)$, ends at $(2n, 0)$, and consists of steps $(1, 1)$ (North-East), called *rises*, and $(1, -1)$ (South-East), called *falls*. We refer to n as the *semilength* of the path. Dyck paths of semilength n are sometimes called Dyck n-paths. A *peak* of a Dyck path is the joint node formed by a rise step immediately followed by a fall step. The *height* of a peak is the y-coordinate of this node.

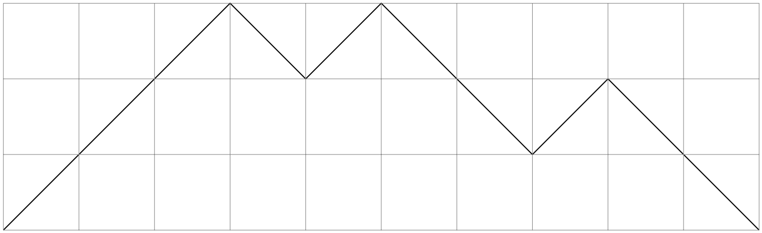
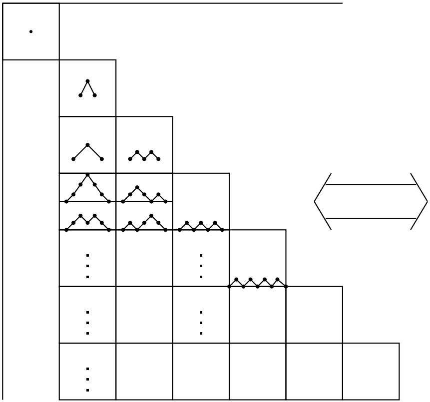


Figure: A Dyck path



	1	2	3	4	5	$\cdots$	k
0	1						
1		1					
2		1	1				
3		2	2	1			
4		5	5	3	1		
5		14	14	9	4	1	
6			$\vdots$		$\vdots$		$\cdots$

Figure: Dyck paths counted per length and per x-axis points

This is the Riordan array $(1, xc(x))$ where

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Its inverse is given by

$$(1, xc(x))^{-1} = (1, x(1 - x))$$

since we have

$$\text{Rev}\{x(1 - x)\} = xc(x).$$

The Riordan array $M = (c(x), xc(x))$ begins

$$M = \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 \\ 14 & 14 & 9 & 4 & 1 & 0 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 \end{array}\right).$$

The (n, k)-th element of this array counts the number of Dyck paths of semilength n which have their first peak at height k.

We find that

$$M^{-1}\overline{M}$$

begins

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

Sequence characterization of Riordan arrays

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & \boxed{3} & \boxed{3} & 1 & 0 & 0 & 0 \\ 1 & 4 & \boxed{6} & 4 & 1 & 0 & 0 \\ 1 & 5 & \boxed{10} & \boxed{10} & 5 & 1 & 0 \\ 1 & 6 & 15 & \boxed{20} & 15 & 6 & 1 \end{array}\right).$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$t_{n,k} = 1 \cdot t_{n-1,k-1} + 1 \cdot t_{n-1,k}$$

$$t_{n,k} = a_0 \cdot t_{n-1,k-1} + a_1 \cdot t_{n-1,k} + a_2 \cdot t_{n-1,k+1} + \cdots .$$

The A sequence

Consider $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$. Here, we have

$$f(x) = \frac{1}{1-x}.$$

Now

$$\bar{f}(x) = \frac{x}{1+x}.$$

Then

$$\frac{x}{\bar{f}(x)} = \frac{x}{\frac{x}{1+x}} = 1 + x.$$

In this case we let

$$A(x) = 1 + x = 1 + 1 \cdot x.$$

This corresponds to

$$t_{n,k} = 1 \cdot t_{n-1,k-1} + 1 \cdot t_{n-1,k}.$$

Now consider

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 & 0 \\ 9 & 12 & 9 & 4 & 1 & 0 & 0 \\ 21 & 30 & 25 & 14 & 5 & 1 & 0 \\ 51 & 76 & 69 & 44 & 20 & 6 & 1 \end{pmatrix}.$$

We can guess that

$$t_{n,k} = 1 \cdot t_{n-1,k-1} + 1 \cdot t_{n-1,k} + 1 \cdot t_{n-1,k+1}.$$

This is the Riordan array

$$\left(\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}, \frac{1-x-\sqrt{1-2x-3x^2}}{2x} \right).$$

We have $f(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$. Solving

$$f(u) = x$$

for u and taking the solution with $u(0) = 0$ gives us

$$\bar{f}(x) = \frac{x}{1+x+x^2}.$$

Then

$$\frac{x}{\bar{f}} = \frac{x}{\frac{x}{1+x+x^2}} = 1 + x + x^2.$$

We get

$$A(x) = 1 + x + x^2 = 1 + 1 \cdot x + 1 \cdot x^2.$$

A *Motzkin path* is a path in the first quadrant which begins at the origin $(0,0)$, ends at $(n,0)$, and consists of steps $(1,1)$ (North-East), called *rises*, and $(1,-1)$ (South-East), called *falls*, and steps $(1,0)$ (East) called *horizontals*. A partial Motzkin path that starts from $(0,0)$ and ends at the point (n,k) (not necessarily on the x-axis) is called a *left factor* of a Motzkin path. See Figure 8.

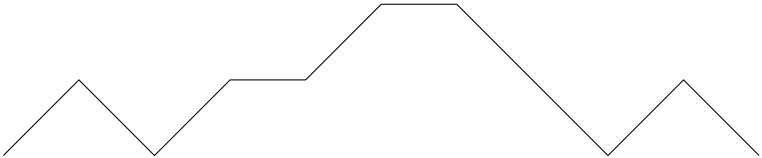
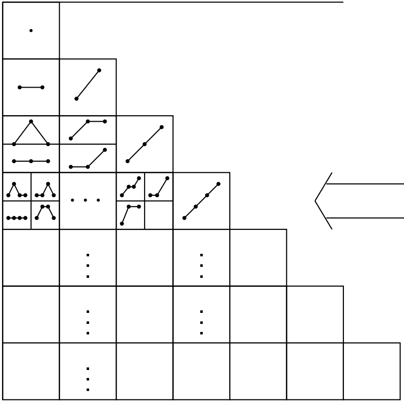


Figure: A Motzkin path



$\Longleftrightarrow$

	1	2	3	4	5	$\cdots$	k
0	1						
1	1	1					
2	2	2	1				
3	4	5	3	1			
4	9	12	9	4	1		
5	21	30	25	14	5	1	
6			$\vdots$		$\vdots$		$\cdots$

Figure: Motzkin left-factors to (n, k)

The Z sequence

The A sequence, where

$$A(x) = \frac{x}{\bar{f}},$$

can characterize the "internal" elements of the Riordan array $(g(x), f(x))$. To characterize the first column elements of $(g(x), f(x))$, we use the Z sequence, where

$$Z(x) = \frac{1}{\bar{f}}\left(1 - \frac{t_{0,0}}{g(\bar{f}(x))}\right).$$

Then we have

$$t_{n,0} = z_0 t_{n-1,0} + z_1 t_{n-1,1} + z_2 t_{n-1,2} + \cdots .$$

The A and the Z sequences

The A sequence operates as

$$\begin{array}{cc} * & \rightarrow \\ & \nwarrow \end{array}$$

while the Z sequence operates as

$$\begin{array}{cc} * & \rightarrow \\ \uparrow & \end{array}$$

The production matrix of an ordinary Riordan array

The pair $(A(x), Z(x))$ is uniquely defined by $(g(x), f(x))$. Thus to the Riordan array $M = (g(x), f(x))$ we can uniquely associate the matrix

$$P = \begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 \\ z_1 & a_1 & a_0 & 0 & 0 & 0 \\ z_2 & a_2 & a_1 & a_0 & 0 & 0 \\ z_3 & a_3 & a_2 & a_1 & a_0 & 0 \\ z_4 & a_4 & a_3 & a_2 & a_1 & a_0 \\ z_5 & a_5 & a_4 & a_3 & a_2 & a_1 \end{pmatrix}.$$

We have

$$P = M^{-1} \cdot \overline{M},$$

where $\overline{M}$ is the matrix M with its top row removed.

When $g_0 = 1$, we have

$$(g, f)^{-1} = \left(1 - \frac{xZ}{A}, \frac{x}{A}\right),$$

and

$$(g, f) = \left(\frac{1}{1 - xZ\left(\mathrm{Rev}\{\frac{x}{A}\}\right)}, \mathrm{Rev}\left\{\frac{x}{A}\right\}\right).$$

Testing for an ordinary Riordan array

The Narayana numbers count the number of Dyck paths from $(0,0)$ to $(2n,0)$ with k peaks.

$$N(n,k) = \frac{1}{k+1}\binom{n+1}{k}\binom{n}{k}.$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 \end{array}\right)$$

This is not an ordinary Riordan array. We can see this easily by calculating the first few rows of its production matrix.

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & 4 & 1 & 0 \\ 0 & -16 & 20 & -10 & 5 & 1 \\ 0 & 130 & -160 & 75 & -20 & 6 \end{array}\right)$$

We note that the generating function of the Narayana triangle may be written as

$$\mathcal{N}(x,y) = \cfrac{1}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-\cdots}}}}.$$

Orthogonal polynomials and ordinary Riordan arrays

Orthogonal polynomials and ordinary Riordan arrays

When $A(x) = 1 + a_1x + a_2x^2$ and $Z(x) = z_0 + z_1x$, we have

$$P = \begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 \\ z_1 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & a_2 & a_1 \end{pmatrix}.$$

Thus P is "tri-diagonal".

Now recall that

$$(g, f)^{-1} = \left(1 - \frac{xZ}{A}, \frac{x}{A}\right).$$

Thus with $A(x) = 1 + a_1 x + a_2 x^2$ and $Z(x) = z_0 + z_1 x$, we have

$$
\begin{aligned}
(g, f)^{-1} &= \left(1 - \frac{x(z_0 + z_1 x)}{1 + a_1 x + a_2 x^2}, \frac{x}{1 + a_1 + a_2 x^2}\right) \\
&= \left(\frac{1 + (a_1 - z_0)x + (a_2 - z_1)x^2}{1 + a_1 x^2 + a_2 x^2}, \frac{x}{1 + a_1 x + a_2 x^2}\right).
\end{aligned}
$$

Orthogonal polynomials

We obtain that when

$$A(x) = 1 + a_1 x + a_2 x^2 \quad \text{and} \quad Z(x) = z_0 + z_1 x,$$

the Riordan array

$$(g, f)^{-1} = \left(\frac{1 + (a_1 - z_0)x + (a_2 - z_1)x^2}{1 + a_1 x + a_2 x^2}, \frac{x}{1 + a_1 x + a_2 x^2} \right)$$

is the coefficient array for the family of orthogonal polynomials $P_n(x)$ given by

$$P_n(x) = (x - a_1)P_{n-1}(x) - a_2 P_{n-2}(x),$$

with

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - z_0, \\ P_2(x) &= x^2 - x(a_1 + z_0) + a_1 z_0 - z_1. \end{aligned}$$

Link to the Chebyshev polynomials

We find that $P_n(x)$ is equal to the following sum of scaled shifted versions of the Chebyshev polynomials of the second kind:

$$(\sqrt{a_2})^n U_n\left(\frac{x-a_1}{2\sqrt{a_2}}\right) - (z_0 - a_1)\,(\sqrt{a_2})^{n-1}\,U_{n-1}\left(\frac{x-a_1}{2\sqrt{a_2}}\right) - (z_1 - a_2)(\sqrt{a_2})^{n-2} U_{n-2}\left(\frac{x-a_1}{2\sqrt{a_2}}\right).$$

Link to the Chebyshev polynomials

Alternatively, the Riordan array

$$\left(\frac{1-\lambda x-\mu x^2}{1+rx^2+sx^2}, \frac{x}{1+rx^2+sx^2}\right)$$

is the coefficient array for

$$(\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right)-\lambda\left(\sqrt{s}\right)^{n-1} U_{n-1}\left(\frac{x-r}{2\sqrt{s}}\right)-\mu(\sqrt{s})^{n-2} U_{n-2}\left(\frac{x-r}{2\sqrt{s}}\right).$$

The Boubaker polynomials

The Boubaker polynomials $B_n(x)$ have coefficient array

$$\left(\frac{1+3x^2}{1+x^2}, \frac{x}{1+x^2}\right).$$

We find that $A(x) = 1 + x^2$, $Z(x) = -2x$.
Thus $a_1 = 0, a_2 = 1, z_0 = 0, z_1 = -2$.
We have

$$B_n(x) = U_n(x/2) - 3U_{n-2}(x/2).$$

The moments of $B_n(x)$ have generating function

$$\cfrac{1}{1+\cfrac{2x^2}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cdots}}}}.$$

Large Schroeder numbers as moments

The large Schroeder numbers are defined by

$$S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k.$$

They have generating function

$$\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right) \cdot c(x) = \frac{1}{1-x} c\left(\frac{x}{(1-x)^2}\right),$$

or

$$\frac{1-x-\sqrt{1-6x+x^2}}{2x}.$$

The ordinary Riordan array

$$\left(\frac{1}{1+2x}, \frac{1}{1+3x+2x^2}\right)$$

is the coefficient array of the orthogonal polynomial family

$$P_n(x) = (x-3)P_{n-1} - 2P_{n-2}(x),$$

with $P_0(x) = 1, P_1(x) = x - 2$, and $P_2(x) = x^2 - 5x + 4$.
Its inverse is given by

$$\left(\frac{1-x-\sqrt{1-6x+x^2}}{2x}, \frac{1-3x-\sqrt{1-6x+x^2}}{4x}\right).$$

Thus the large Schroeder numbers are the moments of this family of orthogonal polynomials. These numbers enumerate Schroeder paths of length n. They also enumerate alternating sign matrices that avoid the pattern 132. We find that

$$S_n = \frac{1}{\pi} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} x^n \frac{\sqrt{-1+6x-x^2}}{2x} \, dx.$$

Recall that we have

$$g(x) = \frac{1}{1-2x-\frac{2x}{1-3x-\frac{2x}{1-3x-\cdots}}}.$$

Generalized orthogonal polynomials defined by Riordan arrays

2-orthogonal polynomials

When

$$A(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3$$

and

$$Z(x) = z_0 + z_1 x + z_2 x^2$$

the Riordan array

$$(g, f)^{-1} = \left(\frac{1 + (a_1 - z_0)x + (a_2 - z_1)x^2 + (a_3 - z_2)x^2}{1 + a_1 x + a_2 x^2 + a_3 x^3}, \frac{x}{1 + a_1 x + a_2 x^2 + a_3 x^3} \right)$$

is the coefficient array of the family of 2-orthogonal polynomials $P_n(x)$ with

$$P_n(x) = (x - a_1)P_{n-1}(x) - a_2 P_{n-2}(x) - a_3 P_{n-3}, \quad n \geq 4.$$

Laurent biorthogonal polynomials

The Riordan array

$$\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$$

is the coefficient array for the biorthogonal polynomials $P_n(x)$ that satisfy

$$P_n(x) = (x - \alpha x)P_{n-1}(x) - \beta x P_{n-2}(x),$$

with

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x - (\alpha + \beta). \end{aligned}$$

Let $(t_{n,k}) = \left(\frac{1+x}{1-x}, \frac{x(1+x)}{1-x}\right)$. Then $t_{n,k}$ is the number of length n words on the alphabet $\{0, 1, 2\}$ with no two consecutive 1's and no two consecutive 2's and having exactly k 0's.

Generalized orthogonal polynomials 1

The Riordan array

$$\left(\frac{1-\beta x}{1+\alpha x+\beta\gamma x^2}, \frac{x(1-\beta x)}{1+\alpha x+\beta\gamma x^2}\right)$$

is the coefficient array of the family of generalized orthogonal polynomials

$$P_n(x) = (x-\alpha)P_{n-1}(x) - \beta(x+\gamma)P_{n-2}(x).$$

Generalized orthogonal polynomials 2

The Riordan array

$$\left(\frac{1+(\alpha-\delta)x}{1+\alpha x+\beta x^2+\gamma x^3}, \frac{x(1-x)}{1+\alpha x+\beta x^2+\gamma x^3}\right)$$

is the coefficient array of the family of generalized orthogonal polynomials $P_n(x)$ that satisfy

$$P_n(x)=(x-\alpha)P_{n-1}(x)-(x+\beta)P_{n-2}(x)-\gamma P_{n-3}(x),$$

with

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x-\delta \\ P_2(x) &= x^2-x(\alpha+\beta+1)+\alpha\delta-\beta. \end{aligned}$$

A path example

Consider the Riordan array

$$(g, f) = \left(\frac{1-x}{1+x^3}, \frac{x(1-x)}{1+x^3}\right).$$

This is the coefficient array of the polynomials that satisfy

$$P_n(x) = xP_{n-1}(x) - xP_{n-2}(x) - P_{n-3}(x),$$

with $P_0(x) = 1$, $P_1(x) = x - 1$, and $P_2(x) = x^2 - 2x$.
The moments of this family of polynomials (the first column elements of $(g, f)^{-1}$) are given by

$$\mu_n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \binom{2n-3k}{n-3k}.$$

These numbers count the number of paths from $(0, 0)$ to (n, n) using steps of three kinds: $(1, 0)$, $(0, 1)$ and $(3, 1)$.

Classical orthogonal polynomials and Riordan arrays

Classical orthogonal polynomials

The classical orthogonal polynomials of mathematical science are the Jacobi, Laguerre and Hermite polynomials, defined by the weights $w_J(x) = (1-x)^\alpha(1+x)^\beta$ on $[-1,1]$, $w_L(x) = x^\alpha e^{-x}$ on $[0,\infty)$, and $w_H(x) = e^{-x^2}$ on $(-\infty,\infty)$, respectively. In particular, these orthogonal polynomials are associated with measures that are absolutely continuous. We have

$$\frac{w_J'(x)}{w_J(x)} = \frac{x(\alpha+\beta)+\alpha-\beta}{x^2-1},$$

$$\frac{w_L'(x)}{w_L(x)} = \frac{\alpha-x}{x},$$

and

$$\frac{w_H'(x)}{w_H(x)} = -2x.$$

A family of orthogonal polynomials $P_n(x)$ is said to be *classical* if the associated measure is absolutely continuous with weight function $w(x)$ satisfying

$$\frac{w'(x)}{w(x)} = \frac{U(x)}{V(x)} = \frac{u_0 + u_1 x}{v_0 + v_1 x + v_2 x^2}.$$

The polynomials $y = P_n(x)$ will then satisfy the differential equation

$$V(x)y'' + (U(x) + V(x))y' - n(u_1 + (n+1)v_2)y = 0.$$

If $\deg(V) > 2$ and/or $\deg(U) > 1$ then we say that the family of polynomials is *semi-classical*.

The Riordan array

$$M=\left(\frac{1+cx+dx^2}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)$$

has moment matrix M^{-1} given by

$$\Big(-\frac{(b-d)\sqrt{1-2ax+x^2(a^2-4b)}+x(a(b+d)-2bc)-b-d}{2(x^2(a^2d-ac(b+d)+b^2+b(c^2-2d)+d^2)+x(c(b+d)-2ad)+d)},$$
$$\frac{1-ax-\sqrt{1-2ax+x^2(a^2-4b)}}{2bx}\Big).$$

The first element of this array is the generating function $\mu(x)$ of the moments of the family of orthogonal polynomials $P_n(x)$. These moments begin

$$1, a-c, a^2-2ac+b+c^2-d, a^3-3a^2c+a(3b+3c^2-3d)-c(2b+c^2-2d),\ldots$$

and their generating function is given by

$$\mu(x)=\cfrac{1}{1-(a-c)x-\cfrac{(b-d)x^2}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-\cdots}}}}.$$

The measure is given by $w(x)dx$ where

$$w(x)=\frac{1}{2\pi}\frac{(b-d)\sqrt{4b-(x-a)^2}}{dx^2+x(c(b+d)-2ad)+a^2d-ac(b+d)+b^2+b(c^2-2d)+d^2}.$$

The ratio $\frac{w'(x)}{w(x)}$ is then given by the expression

$$-\frac{dx^3-3adx^2+x(3a^2d-b^2-b(c^2+6d)-d^2)-a^3d+a(b^2+b(c^2+6d)+d^2)-4bc(b+d)}{((x-a)^2-4b)(dx^2+x(c(b+d)-2ad)+a^2d-ac(b+d)+b^2+b(c^2-2d)+d^2)}.$$

Theorem

The ordinary Riordan array

$$\left(\frac{1+cx+dx^2}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)$$

defines a family of classical orthogonal polynomials in the case that either $c=d=0$ *or* $c=0, d=-b$.

Corollary

When $c = d = 0$, we have

$$w(x) = \frac{1}{2\pi} \frac{\sqrt{4b - (x-a)^2}}{b}$$

on the interval

$$[a - 2\sqrt{b}, a + 2\sqrt{b}].$$

The moments μ_n have integral representation

$$\mu_n = \frac{1}{2\pi} \int_{a-2\sqrt{b}}^{a+2\sqrt{b}} x^n \frac{\sqrt{4b - (x-a)^2}}{b} \, dx.$$

The moments have generating function

$$\mu(x) = \frac{1 - ax - \sqrt{(1-ax)^2 - 4bx^2}}{2bx^2}$$

Corollary

$$\mu(x) = \cfrac{1}{1 - ax - \cfrac{bx^2}{1 - ax - \cfrac{bx^2}{1 - ax - \cfrac{bx^2}{1 - \cdots}}}}.$$

By an application of Lagrange inversion, we obtain

$$\begin{aligned}
\mu_n &= \frac{1}{n+1}[x^n](1+ax+bx^2)^{n+1} \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{j}\binom{j}{n-j}a^{2j-n}b^{n-j} \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{n-k}\binom{n-k}{k}a^{n-2k}b^{k}.
\end{aligned}$$

The moments have Hankel transform

$$h_n = b^{\binom{n+1}{2}}.$$

When $c = d = 0$, the polynomials $P_n(x)$ satisfy the three-term recurrence

$$P_n(x) = (x - a)P_{n-1}(x) - bP_{n-2}(x), \quad n > 1,$$

with $P_0(x) = 1$, $P_1(x) = x - a$.

If $y = P_n(x)$ then y satisfies the differential equation

$$((x - a)^2 - 4b)y'' + 3(x - a)y' - n(n + 2)y = 0.$$

When $c = 0$ and $d = -b$, we have

$$w(x) = \frac{1}{\pi} \frac{1}{\sqrt{4b - (x-a)^2}}$$

on the interval

$$[a - 2\sqrt{b}, a + 2\sqrt{b}].$$

The moments μ_n have integral representation

$$\mu_n = \frac{1}{\pi} \int_{a-2\sqrt{b}}^{a+2\sqrt{b}} x^n \frac{1}{\sqrt{4b - (x-a)^2}} \, dx.$$

The moments have generating function

$$\mu(x) = \frac{1}{\sqrt{(1-ax)^2 - 4bx^2}}$$

given by

$$\mu(x) = \cfrac{1}{1 - ax - \cfrac{2bx^2}{1 - ax - \cfrac{bx^2}{1 - ax - \cfrac{bx^2}{1 - \cdots}}}}.$$

We have the closed form expression for the moments

$$\begin{aligned} \mu_n &= \sum_{i=0}^{n} \binom{n-i}{i} \binom{n-i-1/2}{n-i} (-1)^i (a^2-4b)^i (2a)^{n-2i} \\ &= \frac{1}{4^n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} (a+2\sqrt{b})^k (a-2\sqrt{b})^{n-k}. \end{aligned}$$

The moments have Hankel transform

$$h_n = 2^n b^{\binom{n+1}{2}}.$$

When $c = 0$ and $d = -b$, the polynomials $P_n(x)$ satisfy the three-term recurrence

$$P_n(x) = (x - a)P_{n-1}(x) - bP_{n-2}(x), \quad n > 2,$$

with $P_0(x) = 1$, $P_1(x) = x - a$, and $P_2(x) = (x - a)^2 - b(b + 1)$.

If $y = P_n(x)$ then y satisfies the differential equation

$$((x - a)^2 - 4b)y'' + 3(x - a)y' - n^2 y = 0.$$

Summary

When $c = d = 0$, we have

$$w(x) = \frac{1}{2\pi}\frac{\sqrt{4b-(x-a)^2}}{b}, \quad \text{on } [a-2\sqrt{b}, a+2\sqrt{b}]$$

$$P_n(x) = (x-a)P_{n-1}(x) - bP_{n-2}(x), \quad n > 1,$$

with $P_0(x) = 1, P_1(x) = x - a$.

$$((x-a)^2 - 4b)y'' + 3(x-a)y' - n(n+2)y = 0.$$

When $c = 0$ and $d = -b$, we have

$$w(x) = \frac{1}{\pi}\frac{1}{\sqrt{4b-(x-a)^2}}, \quad \text{on } [a-2\sqrt{b}, a+2\sqrt{b}]$$

$$P_n(x) = (x-a)P_{n-1}(x) - bP_{n-2}(x), \quad n > 2,$$

with $P_0(x) = 1$, $P_1(x) = x - a$, and $P_2(x) = (x-a)^2 - b(b+1)$.

$$((x-a)^2 - 4b)y'' + 3(x-a)y' - n^2 y = 0.$$

A non-classical example

For Riordan arrays of the type

$$\left(\frac{1+rx^2}{1+x^2}, \frac{x}{1+x^2}\right),$$

we have

$$P_n(x;r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n-(r+1)k}{n-k} (-1)^k x^{n-2k}.$$

We have the following integral representation of the moment sequence $\mu_n(r)$.

$$\mu_n(r) = \frac{-1}{\pi} \int_2^2 x^n \frac{\sqrt{4-x^2}(r-1)}{2(rx^2+(r-1)^2)}\, dx + \frac{r+1}{2r}\left(-\frac{r-1}{\sqrt{-r}}\right)^n + \frac{r+1}{2r}\left(\frac{r-1}{\sqrt{-r}}\right)$$

This shows that in this case, the measure defining the orthogonal polynomials is no longer absolutely continuous, but it reflects the zeros of the denominator term $rx^2+(r-1)^2$.

The Exponential Riordan Group

Exponential Riordan arrays

An exponential Riordan array is defined by two power series of exponential type

$$g(x) = g_0 + g_1 \frac{x}{1!} + g_2 \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!},$$

and

$$g(x) = 0 + f_1 \frac{x}{1!} + f_2 \frac{x^2}{2!} + \cdots = \sum_{n=1}^{\infty} f_n \frac{x^n}{n!}.$$

We will generally take $g_0 = 1$ and $f_1 = 1$. The exponential Riordan array associated to the datum $(g(x), f(x))$ is defined to be the invertible lower-triangular matrix with general (n, k)-th element

$$t_{n,k} = \frac{n!}{k!}[x^n]g(x)f(x)^k.$$

We will denote this matrix by $[g(x), f(x)]$.

The Identity matrix

Consider the exponential Riordan array $[1, x]$.
We have

$$\begin{aligned} t_{n,k} &= \frac{n!}{k!}[x^n]1.x^k \\ &= \frac{n!}{k!}[x^{n-k}]\,1 \\ &= \frac{n!}{k!}[x^{n-k}]\,x^0 \\ &= \frac{n!}{k!}\delta_{n-k} \\ &= 1, \text{if} \quad n=k, \quad \text{else} \quad 0. \end{aligned}$$

Thus $[1, x]$ is the identity matrix.

The Binomial matrix

Consider the exponential Riordan array $[e^x, x]$. We have

$$\begin{aligned} t_{n,k} &= [x^n]\, e^x x^k \\ &= \frac{n!}{k!}[x^{n-k}]\, e^x \\ &= \frac{n!}{k!}[x^{n-k}] \sum_{i=0}^{\infty} \frac{x^i}{i!} \\ &= \frac{n!}{k!} \frac{1}{(n-k)!} \\ &= \binom{n}{k}. \end{aligned}$$

Thus

$$[e^x, x] = \left(\binom{n}{k}\right)$$

is the binomial matrix.

The Exponential Riordan group

The set of exponential Riordan arrays is a group for the operation of matrix multiplication. We have

$$[g(x), f(x)] \cdot [u(x), v(x)] = [g(x)u(f(x)), v(f(x))]$$

and

$$[g(x), f(x)]^{-1} = \left[\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right].$$

The matrix $[1, x]$ is the identity of the group.

Row sums

The power series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the exponential generating function of the sequence

$$1, 1, 1, 1, 1, \ldots.$$

That is, $n![x^n]e^x = 1$ for all n.

The row sums of the exponential Riordan array $[g(x), f(x)]$ then have generating function

$$[g(x), f(x)] \cdot e^x = g(x)e^{f(x)} = g(x) \exp(f(x)).$$

Stirling numbers of the second kind example

Consider the exponential Riordan array

$$[e^x, e^x - 1].$$

We shall calculate its (n, k)-th element

$$t_{n,k} = \frac{n!}{k!}[x^n]e^x(e^x - 1)^k.$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 & 0 \\ 1 & 63 & 301 & 350 & 140 & 21 & 1 \end{array}\right).$$

$$
\begin{aligned}
t_{n,k} &= \frac{n!}{k!}\sum_{i=0}^{\infty}\frac{x^i}{i!}\sum_{j=0}^{k}\binom{k}{j}e^{jx}(-1)^{k-j}\\
&= \frac{n!}{k!}\sum_{i=0}^{\infty}\frac{x^i}{i!}\sum_{j=0}^{k}\binom{k}{j}\sum_{\ell=0}^{\infty}\frac{j^{\ell}x^{\ell}}{\ell!}(-1)^{k-j}\\
&= \frac{n!}{k!}\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}^{k}\binom{k}{j}\frac{j^{n-i}}{(n-i)!}(-1)^{k-j}\\
&= \frac{1}{k!}\sum_{i=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{n!}{i!(n-i)!}j^{n-i}(-1)^{k-j}\\
&= \frac{1}{k!}\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{k}\binom{k}{j}j^{n-i}(-1)^{k-j}.
\end{aligned}
$$

This array has a production matrix that begins

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{array}\right).$$

FTRA for exponential Riordan arrays

We have used the Fundamental Theorem of exponential Riordan arrays, which states that

$$\boxed{[\mathbf{g(x)}, \mathbf{f(x)}] \cdot \mathbf{A(x)} = \mathbf{g(x)A(f(x))}.}$$

The A and the Z sequences of $[g(x), f(x)]$

We wish to associate two exponential power series A and Z to the Riordan array $M = [g(x), f(x)]$ so that some combination of A and Z will generate the production matrix

$$P = M^{-1}\overline{M}.$$

We find the following.
If

$$\boxed{A(x) = f'(\bar{f}(x)) \quad \text{and} \quad Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))},}$$

then the expression in x and y given by

$$e^{xy}(Z(x) + yA(x)) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} p_{n,j} y^j \frac{x^n}{n!}$$

generates the production matrix P.

Production matrix example 1

We consider the exponential Riordan array

$$\left[\frac{1}{1-x}, \frac{x}{1-x}\right].$$

We have

$$\begin{aligned} n![x^n]\frac{1}{1-x} &= n![x^n]\sum_{i=0}^{\infty} x^i \\ &= n! \cdot 1 = n! \end{aligned}$$

Thus the first column of this array is given by $n!$ or

$$1, 1, 2, 6, 24, 120, \ldots.$$

$$\begin{aligned} t_{n,k} = \frac{n!}{k!}[x^n]\frac{1}{1-x}\left(\frac{x}{1-x}\right)^k &= \frac{n!}{k!}[x^{n-k}](1-x)^{-(k+1)} \\ &= \frac{n!}{k!}\binom{n}{k}. \end{aligned}$$

Production matrix example 2

$$f(x) = \frac{x}{1-x} \Longrightarrow f'(x) = \frac{1}{(1-x)^2}.$$

Also, $\bar{f}(x) = \frac{x}{1+x}$. Hence

$$A(x) = f'(\bar{f}(x)) = \frac{1}{(1-\frac{x}{1+x})^2} = (1+x)^2.$$

We have $g(x) = \frac{1}{1-x}$ and so $g'(x) = \frac{1}{(1-x)^2}$. Then

$$Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = 1 + x.$$

Thus the production matrix is generated by

$$e^{xy}(1 + x + y(1 + 2x + x^2)).$$

Production matrix example 3

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 4 & 5 & 1 & 0 & 0 \\ 0 & 0 & 9 & 7 & 1 & 0 \\ 0 & 0 & 0 & 16 & 9 & 1 \\ 0 & 0 & 0 & 0 & 25 & 11 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ {}^{1}2 & {}^{1}4 & 1 & 0 & 0 & 0 & 0 \\ \boxed{6} & 18 & 9 & 1 & 0 & 0 & 0 \\ {}^{1}24 & {}^{3}96 & {}^{4}72 & 16 & 1 & 0 & 0 \\ 120 & \boxed{600} & 600 & {}^{1}200 & {}^{9}25 & {}^{25}1 & 0 \\ 720 & 4320 & 5400 & 2400 & \boxed{450} & 36 & 1 \end{pmatrix}$$

$[g, f]$ in terms of A and Z

We have

$$[g(x), f(x)]^{-1} = \left[\frac{1}{e^{\int_0^x \frac{Z(t)}{A(t)}\,dt}}, \int_0^x \frac{1}{A(t)}\,dt\right].$$

and

$$[g(x), f(x)] = \left[e^{\int_0^x Z(\text{Rev}\left(\int_0^t \frac{dt}{A(t)}\right))dt}, \text{Rev}\left(\int_0^x \frac{dt}{A(t)}\right)\right].$$

Alternatively, we can write

$$[g(x), f(x)] = \left[e^{\int_0^{\text{Rev}\left(\int_0^x \frac{dt}{A(t)}\right)} \frac{Z(t)}{A(t)}\,dt}, \text{Rev}\left(\int_0^x \frac{dt}{A(t)}\right)\right].$$

A and Z for $[g, f]^{-1}$

Let the A sequence and the Z sequence of $[g, f]^{-1}$ be denoted by $A^*(x)$ and $Z^*(x)$. Let

$$[u, v] = [g, f]^{-1} = \left[e^{\int_0^x \frac{Z(t)}{A(t)}\, dt}, \int_0^x \frac{1}{A(t)}\, dt \right].$$

Then

$$A^*(x) = v'(\bar{v}) = \frac{1}{A\left(\mathrm{Rev}\{\int_0^x \frac{1}{A(t)}\, dt\}\right)}.$$

Also,

$$Z^*(x) = \frac{u'(\bar{v})}{u(\bar{v})} = -\frac{Z\left(\mathrm{Rev}\{\int_0^x \frac{1}{A(t)}\, dt\}\right)}{A\left(\mathrm{Rev}\{\int_0^x \frac{1}{A(t)}\, dt\}\right)}.$$

Let M be the matrix with

$$A(x) = \frac{1}{1-x}, \qquad Z(x) = \frac{1}{1-x}.$$

We have

$$\int_0^x \frac{1}{A(t)}\,dt = \int_0^x (1-t)\,dt = x - \frac{x^2}{2}.$$

Now

$$\text{Rev}\{x - \frac{x^2}{2}\} = 1 - \sqrt{1-2x}$$

and hence

$$A^*(x) = \frac{1}{A(1-\sqrt{1-2x})} = \sqrt{1-2x}.$$

In this case, $A(x) = Z(x)$ implies that $Z^*(x) = -1$.

Exponential Riordan arrays and polynomial families

Orthogonal polynomials and exponential Riordan arrays

If

$$A(x) = 1 + \alpha x + \beta x^2$$

and

$$Z(x) = \gamma + \delta x$$

then the production matrix is tri-diagonal and the inverse array $[g, f]^{-1}$ will be the coefficient array of a family of polynomials. In this case we have

$$[g, f]^{-1} = \left[\frac{1}{e^{\int_0^x \frac{\gamma + \delta t}{1 + \alpha t + \beta t^2}\, dt}}, \int_0^x \frac{1}{1 + \alpha t + \beta t^2}\, dt \right]$$

is the coefficient array for the polynomial family $P_n(x)$ where

$$P_n(x) = (x - (\alpha + (n-1)\gamma) P_{n-1}(x) - ((n-1)\beta + (n-1)(n-2)\delta) P_{n-2},$$

with $P_0(x) = 1$, $P_1(x) = x - \alpha$.

Laguerre polynomials

We let $A(x) = 1 + 2x + x^2$ and $Z(x) = 1 + x$.

$$\int_0^x \frac{1}{1+2t+t^2}\, dt = \frac{x}{1+x}.$$

$$\int_0^x \frac{1+t}{1+2t+t^2}\, dt = \ln(1+x).$$

Then

$$e^{-\ln(1+x)} = \frac{1}{1+x}.$$

Thus

$$[g, f]^{-1} = \left[\frac{1}{1+x}, \frac{x}{1+x}\right]$$

is the coefficient array of the family of orthogonal polynomials

$$P_n(x) = (x - (2n-1))P_{n-1}(x) - (n-1)^2 P_{n-2}(x).$$

Modified Hermite polynomials

Consider the exponential Riordan array

$$\left[e^{\frac{x^2}{2}}, x\right].$$

We have $f(x) = x \Longrightarrow f'(x) = 1$, and thus $A(x) = 1$. Also, $g(x) = e^{\frac{x^2}{2}} \Longrightarrow g'(x) = xg(x)$, and thus $Z(x) = x$. The production matrix is then

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{array}\right).$$

Modified Hermite polynomials

The production matrix is generated by

$$e^{xy}(x+y).$$

The associated orthogonal polynomials obey

$$P_n(x) = xP_{n-1}(x) - (n-1)P_{n-2}(x),$$

with $P_0(x) = 1$, $P_1(x) = x$.

2-orthogonal polynomials

We let $A(x) = 1 + 3x + 3x^2 + x^3$ and $Z(x) = 1 + x + x^2$.

$$\int_0^x \frac{1}{(1+t)^3}\,dt = \frac{x(x+2)}{2(1+x)^2}.$$

$$\int_0^x \frac{1+t+t^2}{(1+t)^3}\,dt = \ln(1+x) - \frac{x^2}{2(1+x)^2}.$$

Then

$$[g, f]^{-1} = \left[\frac{e^{-\frac{x^2}{2(1+x)^2}}}{1+x}, \frac{x(x+2)}{2(1+x)^2}\right]$$

is the coefficient array of the family of 2-orthogonal polynomials

$$P_n(x) = (x-(3n-2))P_{n-1}(x) - 3(n-1)^2 P_{n-2}(x) - (n-1)(n-2)^2 P_{n-3}(x),$$

$$P_0(x) = 1, P_1(x) = x - 1, P_2(x) = x^2 - 5x + 1.$$

We have

$$[g,f] = \left[\frac{e^{1-x-\sqrt{1-2x}}}{\sqrt{1-2x}}, 1-\sqrt{1-2x}\right].$$

Note that the array

$$\left[\frac{1}{\sqrt{1-2x}}, 1-\sqrt{1-2x}\right]$$

is the coefficient array of the Bessel polynomials.

We have

$$[g, f] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 21 & 36 & 12 & 1 & 0 & 0 & 0 & 0 \\ 153 & 321 & 147 & 22 & 1 & 0 & 0 & 0 \\ 1410 & 3465 & 1980 & 415 & 35 & 1 & 0 & 0 \\ 15765 & 44010 & 29790 & 7890 & 945 & 51 & 1 & 0 \\ 207375 & 643965 & 499590 & 158130 & 24150 & 1869 & 70 & 1 \end{pmatrix}$$

where the first column elements

$$1, 1, 4, 21, 153, \ldots$$

can be considered the moments of the family of polynomials (though we should pair this with the second column - see later).

The production matrix is 4-diagonal.

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 & 0 \\ 2 & 12 & 7 & 1 & 0 & 0 & 0 \\ 0 & 12 & 27 & 10 & 1 & 0 & 0 \\ 0 & 0 & 36 & 48 & 13 & 1 & 0 \\ 0 & 0 & 0 & 80 & 75 & 16 & 1 \\ 0 & 0 & 0 & 0 & 150 & 108 & 19 \end{pmatrix}.$$

Generating function for the moments

The moments have the following continued fraction expression for the ordinary generating function.

$$\cfrac{1}{1-x-\cfrac{3x^2}{1-4x-\frac{12x^2}{(.)}-\frac{12x^3}{(.)(.)}}-\cfrac{2x^3}{(1-4x-\frac{12x^2}{(.)}-\frac{12x^3}{(.)(.)})(1-7x-\frac{27x^2}{(.)}-\frac{36x^3}{(.)(.)})}}.$$

2-Hankel transform

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 21 & 36 & 12 & 1 & 0 & 0 & 0 & 0 \\ 153 & 321 & 147 & 22 & 1 & 0 & 0 & 0 \\ 1410 & 3465 & 1980 & 415 & 35 & 1 & 0 & 0 \\ 15765 & 44010 & 29790 & 7890 & 945 & 51 & 1 & 0 \\ 207375 & 643965 & 499590 & 158130 & 24150 & 1869 & 70 & 1 \end{pmatrix}$$

Consider the two sequences a_n (the first column) and b_n (the sum of first and second column) as follows:

$$1, 1, 4, 21, 153, 1410, 15765, 207375, \ldots.$$

$$1, 2, 9, 57, 474, 4875, 59775, 851340, \ldots.$$

2-Hankel transform

We define the 2 Hankel transform of (a_n, b_n) to be

$$h_n = \begin{cases} |a_{i+j-\lfloor \frac{i}{2} \rfloor}|_{0 \le i,j \le n} & \text{if } i \text{ is even} \\ |b_{i+j-\lfloor \frac{i+1}{2} \rfloor}|_{0 \le i,j \le n} & \text{if } i \text{ is odd} \end{cases}$$

Then

$$h_n = \prod_{k=0}^{n} \gamma_k^{\lfloor \frac{n-k}{2} \rfloor}$$

where in this case we have

$$\gamma_n = (n+1)^2(n+2).$$

That is, γ_n is the sequence

$$2, 12, 36, 80, 150, 252, 392, \ldots.$$

We can recover the associated 2-orthogonal polynomials $P_n(x)$ using determinants as follows. We have

$$P_n(x) = \frac{h_n(x)}{h_{n-1}}$$

where $h_n(x)$ is the same as the determinant h_n, except that the last row is given by $1, x, x^2, \ldots$.

Another example of 2-orthogonality

The exponential Riordan array

$$M = \left[e^{-\tanh(x)}, \tanh(x)\right]$$

has production matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & -2 & -1 & 1 & 0 & 0 \\ 0 & 6 & -6 & -1 & 1 & 0 \\ 0 & 0 & 12 & -12 & -1 & 1 \\ 0 & 0 & 0 & 20 & -20 & -1 \end{pmatrix}.$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -3 & 1 & 0 & 0 \\ -7 & 12 & -2 & -4 & 1 & 0 \\ 3 & -39 & 50 & -10 & -5 & 1 \end{pmatrix}.$$

Let a_n and b_n be the sequences

$$1, -1, 1, 1, -7, 3, 97, -275, -2063, 15015, 53409, \ldots,$$

$$1, 0, -1, 2, 5, -36, -21, 958, -1527, -35816, 169655, \ldots.$$

The 2-Hankel transform of (a_n, b_n) is given by

$$h_n = \prod_{k=0}^{n} ((k+1)(k+2))^{\lfloor \frac{n-k}{2} \rfloor} = \prod_{k=0}^{n} k!.$$

$$M^{-1} = \left[e^{-\tanh(x)}, \tanh(x)\right]^{-1} = \left[e^x, \ln\sqrt{\frac{1+x}{1-x}}\right]$$

is the coefficient array of a family $P_n(x)$ of 2-orthogonal polynomials. We have

$$M^{-1} = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 5 & 3 & 1 & 0 & 0 \\ 1 & 12 & 14 & 4 & 1 & 0 \\ 1 & 49 & 50 & 30 & 5 & 1 \end{array}\right).$$

$$P_n(x) = (x+1)P_{n-1}(x) + (n-1)(n-2)P_{n-2}(x) - (n-1)(n-2)P_{n-3}(x).$$

An example related to the Stirling numbers

We let

$$A = \left[e^x e^{\frac{(e^x-1)^2}{2}}, e^x - 1 \right] = \left[e^{\frac{(e^x-1)^2}{2}}, x \right] \cdot [e^x, e^x - 1] \,.$$

The array A begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 7 & 10 & 6 & 1 & 0 & 0 & 0 \\ 29 & 45 & 31 & 10 & 1 & 0 & 0 \\ 136 & 241 & 180 & 75 & 15 & 1 & 0 \\ 737 & 1428 & 1186 & 560 & 155 & 21 & 1 \end{array}\right).$$

The production matrix of A begins

$$P_A = \left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 6 & 3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 12 & 4 & 5 & 1 & 0 \\ 0 & 0 & 0 & 20 & 5 & 6 & 1 \\ 0 & 0 & 0 & 0 & 30 & 6 & 7 \end{array}\right),$$

with

$$A(x) = 1 + x \qquad Z(x) = 1 + x + x^2.$$

$$Q_n(x) = (x-n)Q_{n-1}(x) - (n-1)Q_{n-2}(x) - (n-1)(n-2)Q_{n-3}(x),$$

$Q_0(x) = 1, Q_1(x) = x - 1, Q_2(x) = x^2 - 3x + 1.$

We let

$$B = \left[e^x e^{\frac{e^{2x}-1}{2}}, \frac{e^{2x}-1}{2}\right].$$

The array B begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 & 0 & 0 \\ 24 & 34 & 12 & 1 & 0 & 0 & 0 \\ 116 & 208 & 112 & 20 & 1 & 0 & 0 \\ 648 & 1396 & 1000 & 280 & 30 & 1 & 0 \\ 4088 & 10232 & 9076 & 3480 & 590 & 42 & 1 \end{array}\right).$$

The production array of B begins

$$P_B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 8 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & 10 & 12 & 1 \\ 0 & 0 & 0 & 0 & 0 & 12 & 14 \end{pmatrix},$$

$$A(x) = 1 + 2x \qquad Z(x) = 2 + 2x.$$

$$P_n(x) = (x - 2n)P_{n-1}(x) - 2(n-1)P_{n-2}(x),$$

$P_0(x) = 1, P_1(x) + x - 2.$

We have the following product

$$A^{-1}B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 10 & 21 & 10 & 1 & 0 & 0 \\ 1 & 15 & 55 & 55 & 15 & 1 & 0 \\ 1 & 21 & 120 & 215 & 120 & 21 & 1 \end{pmatrix}.$$

This is the array

$$\left[e^x, x + \frac{x^2}{2}\right],$$

with

$$A(x) = \sqrt{1+2x} \qquad Z(x) = 1,$$

the number of k-matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$.

We note that the exponential Riordan array

$$A = \left[e^x e^{\frac{(e^x-1)^3}{3}}, e^x - 1\right] = \left[e^{\frac{(e^x-1)^3}{3}}, x\right] \cdot [e^x, e^x - 1]$$

has a 5-diagonal production matrix.

$$\left(\begin{array}{cccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 24 & 12 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 60 & 20 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 120 & 30 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 210 & 42 & 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 336 & 56 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 504 & 72 & 0 & 9 \end{array}\right)$$

Its inverse is then the coefficient array of a family of 3-orthogonal polynomials.

We have

$$\left[e^x e^{\frac{(e^x-1)^3}{3}}, e^x-1\right]^{-1} = \left[e^{-\frac{x^3}{3}}, \ln(1+x)\right].$$

In general, the production matrix of

$$\left[e^x e^{\frac{(e^x-1)^r}{r}}, e^x-1\right]$$

is generated by

$$e^{xy}(x^{r-1}+x^r+y(1+x)).$$

This production matrix is thus $r+2$ diagonal. The inverse exponential array

$$\left[e^{-\frac{x^r}{r}}, \ln(1+x)\right]$$

is the coefficient array of a family of r-orthogonal polynomials.

An interesting Riordan array

The exponential Riordan array

$$M = \left[\frac{e^{-x}}{(1-x)^3}, \frac{x}{1-x}\right]$$

has a production matrix that begins

$$\left(\begin{array}{ccccccc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 15 & 8 & 1 & 0 & 0 \\ 0 & 0 & 0 & 24 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & 35 & 12 & 1 \\ 0 & 0 & 0 & 0 & 0 & 48 & 14 \end{array}\right).$$

Thus its inverse M^{-1} is the coefficient array of a family of orthogonal polynomials.

The inverse array

$$M^{-1} = \left[\frac{e^{\frac{x}{1+x}}}{(1+x)^3}, \frac{x}{1+x}\right]$$

has a production matrix that begins

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 0 \\ 2 & 4 & -6 & 1 & 0 & 0 & 0 \\ 0 & 6 & 9 & -8 & 1 & 0 & 0 \\ 0 & 0 & 12 & 16 & -10 & 1 & 0 \\ 0 & 0 & 0 & 20 & 25 & -12 & 1 \\ 0 & 0 & 0 & 0 & 30 & 36 & -14 \end{pmatrix}.$$

Hence its inverse, or M, is the coefficient array of a family of 2-orthogonal polynomials. The corresponding 2-Hankel transform is equal to $\prod_{k=0}^{n} k!$

Riordan arrays and elliptic functions

Figure: Karl Jacobi (1804-1851) & Karl Weierstrass (1815-1897)

Jacobi Elliptic functions

We define $sn(x,k)$ by

$$\mathrm{sn}(x,k) = \mathsf{Rev} \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

and then we define $\mathrm{cn}(x,k)$ and $dn(x,k)$ as follows.

$$\mathrm{cn}^2(x) + \mathrm{sn}^2(x) = 1$$

$$\mathrm{dn}^2(x) + k^2\,\mathrm{sn}^2(x) = 1.$$

We then have

$$\mathrm{sn}'(x) = \mathrm{cn}(x)\,\mathrm{dn}(x).$$

$$\mathrm{cn}'(x) = -\,\mathrm{sn}(x)\,\mathrm{dn}(x).$$

$$\mathrm{dn}'(x) = -k^2\,\mathrm{sn}(x)\,\mathrm{cn}(x).$$

We have the following special values.

$$\mathrm{dn}(x,0)=1.$$

$$\mathrm{sn}(x,0)=\sin(x).$$

$$\mathrm{cn}(x,0)=\cos(x).$$

$$\mathrm{sn}(x,1)=\tanh(x).$$

$$\mathrm{cn}(x,1)=\mathrm{dn}(x,1)=\mathrm{sech}(x).$$

Consider the following Riordan array

$$\left(1, \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right)^{-1} = (1, sn(x)).$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6}\left(k^2+1\right) & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\left(k^2+1\right) & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{40}\left(3k^4+2k^2+3\right) & 0 & \frac{1}{2}\left(k^2+1\right) & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{45}\left(8k^4+7k^2+8\right) & 0 & \frac{2}{3}\left(k^2+1\right) & 0 & 1 \end{array}\right)^{-1}$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6}\left(-k^2-1\right) & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\left(-k^2-1\right) & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{120}\left(k^4+14k^2+1\right) & 0 & \frac{1}{2}\left(-k^2-1\right) & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{45}\left(2k^4+13k^2+2\right) & 0 & -\frac{2}{3}\left(k^2+1\right) & 0 & 1 \end{array}\right).$$

We see that $\mathrm{sn}(x,k)$ expands to give the sequence

$$0, 1, 0, -\frac{k^2+1}{6}, 0, \frac{k^4+14k^2+1}{120}, 0, -\frac{k^6+135k^4+135k^2+1}{5040}, 0,$$

$$\frac{k^8+1228k^6+5478k^4+1228k^2+1}{362880}, 0, \ldots.$$

Ignoring signs, the numerator coefficient array for this sequence of polynomials begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 14 & 1 & 0 & 0 & 0 & 0 \\ 1 & 135 & 135 & 1 & 0 & 0 & 0 \\ 1 & 1228 & 5478 & 1228 & 1 & 0 & 0 \\ 1 & 11069 & 165826 & 165826 & 11069 & 1 & 0 \end{array}\right)$$

The bivariate generating function for this triangle can be expressed as a continued fraction.

$$\cfrac{1}{1-x-xy-\cfrac{12x^2y}{1-9x-9xy-\cfrac{240x^2y}{1-25x-25xy-\cfrac{1260x^2y}{1-49x-49xy-\cfrac{4032x^2y}{1-\cdots}}}}}.$$

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{12x}{1+\frac{3}{1}x-\cfrac{9xy}{1-\cfrac{\frac{240}{9}x}{1+\frac{5}{3}x-\cfrac{25xy}{1-\cfrac{\frac{1260}{25}x}{1+\frac{7}{5}x-\cfrac{49xy}{1-\cdots}}}}}}}}.$$

$$1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, \ldots$$

where the numbers are taken in groups of 4 ($1 \cdot 2 \cdot 2 \cdot 3 = 12$ etc) for the "β" coefficients, and the odd numbers two by two for the "α" coefficients ($1 \cdot 1 = 1, 3 \cdot 3 = 9$ etc).

Elliptic functions

We recall that

$$[g(x), f(x)] = \left[e^{\int_0^{\mathrm{Rev}\left(\int_0^x \frac{dt}{A(t)}\right)} \frac{Z(t)}{A(t)}\,dt}, \mathrm{Rev}\left(\int_0^x \frac{dt}{A(t)}\right)\right].$$

Now let

$$A(t) = \sqrt{(1-t^2)(1-k^2t^2)}.$$

Then

$$\mathrm{Rev}\left(\int_0^x \frac{dt}{A(t)}\right) = \mathrm{Rev}\left(\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right) = \mathrm{sn}(x,k),$$

the elliptic sine function.

Furthermore, if

$$Z(t) = -\frac{t\sqrt{1-k^2t^2}}{\sqrt{1-t^2}},$$

then

$$\frac{Z(t)}{A(t)} = \frac{-t}{1-t^2}$$

and

$$\int_0^{\mathrm{sn}(x,k)} \frac{Z(t)}{A(t)}\,dt = \left[\sqrt{1-x^2}\right]_0^{\mathrm{sn}(x,k)} = \mathrm{cn}(x,k).$$

Thus

$$A(t) = A(t) = \sqrt{(1-t^2)(1-k^2t^2)} \quad \text{and} \quad Z(t) = -\frac{t\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

results in

$$M = [g(x), f(x)] = [\mathrm{cn}(x,k), \mathrm{sn}(x,k)].$$

Elliptic functions

To calculate the inverse array of $[\mathrm{cn}(x), \mathrm{sn}(x)]$, we have

$$\frac{1}{g(\bar{f}(x))} = \frac{1}{cn(sn^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Thus

$$M^{-1} = [(n), \mathrm{sn}(x)]^{-1} = \left[\frac{1}{\sqrt{1-x^2}}, \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right].$$

We get

$$A_{M^{-1}} = \frac{1}{\mathrm{cn}(x)\,\mathrm{dn}(x)}, \qquad Z_{M^{-1}} = \frac{\mathrm{sn}(x)}{\mathrm{cn}(x)^2}.$$

Consideration of the case

$$A(x) = \mathrm{cn}(x,k), \qquad Z(x) = \mathrm{cn}(x,k)$$

leads to some interesting results. We have

$$\begin{aligned}\int_0^x \frac{dt}{A(t)} &= \int_0^x \frac{dt}{\mathrm{cn}(t,k)} \\ &= \frac{1}{\sqrt{1-k}} \log\left(\frac{\mathrm{dn}(x)+\sqrt{1-k}\,\mathrm{sn}(x)}{\mathrm{cn}(x)}\right).\end{aligned}$$

Since $A(x) = Z(x)$, we find that the inverse matrix $[g,f]^{-1}$ is given by

$$\left[e^{-x}, \frac{1}{\sqrt{1-k}} \log\left(\frac{\mathrm{dn}(x)+\sqrt{1-k}\,\mathrm{sn}(x)}{\mathrm{cn}(x)}\right)\right].$$

The production matrix of this inverse array begins

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -3 & 1 & 1 & 0 & 0 \\ 4k+1 & 4k+1 & -6 & -6 & 1 & 1 & 0 \\ 0 & 20k+5 & 20k+5 & -10 & -10 & 1 & 1 \\ -16k^2-44k-1 & -16k^2-44k-1 & 60k+15 & 60k+15 & -15 & -15 & 1 \end{array}\right).$$

The production matrix of $[g, f]$ begins

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 1 & 0 \\ 0 & 1-4\cdot k & 0 & 6 & -1 & 1 \\ 0 & 0 & 5\cdot(1-4\cdot k) & 0 & 10 & -1 \end{pmatrix}.$$

For $k = 1/4$, $P_{M^{-1}}$ takes on the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -3 & 1 & 1 & 0 & 0 \\ 2 & 2 & -6 & -6 & 1 & 1 & 0 \\ 0 & 10 & 10 & -10 & -10 & 1 & 1 \\ -13 & -13 & 30 & 30 & -15 & -15 & 1 \end{pmatrix}.$$

This prompts us to look at $\mathrm{cn}(x, 1/4)$. As a generating function, this expands to give the sequence

$$1, 0, -1, 0, 2, 0, -13, 0, 161, 0, -3094, 0, 87773, \ldots.$$

We have

$$\mathrm{cn}(x, 1/4) = \cfrac{1}{1+\cfrac{x^2}{1+\cfrac{x^2}{1+\cfrac{9x^2}{1+\cfrac{4x^2}{1+\cfrac{25x^2}{1+\cdots}}}}}}.$$

The coefficients are the squares of the interleaving sequence of odd numbers with the natural numbers

$$1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, 15, 8, 17, 9, \ldots$$

or $a_n = \frac{2(n+1)}{3-(-1)^n}$.

We deduce that $\mathrm{cn}(x, 1/4)$ is the generating function of the moments of the family of orthogonal polynomials defined by

$$P_n(x) = xP_{n-1}(x) + \left(\frac{2(n-1)}{3-(-1)^n}\right)^2 P_{n-2}(x),$$

with

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x. \end{aligned}$$

The Hankel transform of these moments is given by

$$h_n = (-1)^{\binom{n+1}{2}} \prod_{k=0}^{n} \left(\frac{2(k+1)}{3-(-1)^k}\right)^{2(n-k)}.$$

This sequence begins

$$1, -1, -1, 9, 324, -291600, -2361960000, 937461924000000, \ldots$$

The coefficient array for $P_n(x)$ begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 11 & 0 & 1 & 0 & 0 \\ 0 & 17 & 0 & 15 & 0 & 1 & 0 \\ 225 & 0 & 292 & 0 & 40 & 0 & 1 \end{array}\right).$$

The first column elements $P_n(0)$

$$1, 0, 1, 0, 9, 0, 225, 0, 11025, 0, 893025, \ldots$$

have e.g.f. given by $\frac{1}{\sqrt{1-x^2}}$ and they count the number of permutations of in S_{2n} whose cycles are all even. Tao has shown that

$$P_n(0) = (1+x^2)^{\frac{n+1}{2}} \frac{d^n}{dx^n}(1+x^2)^{\frac{n-1}{2}}.$$

The function $\mathrm{dn}(2x, 1/4)$ expands to give the sequence

$$1, 0, -1, 0, 17, 0, -433, 0, 20321, 0, -1584289, 0, 179967473, 0, -28151779537, \ldots.$$

We have

$$\mathrm{dn}(2x, 1/4) = \cfrac{1}{1+\cfrac{x^2}{1+\cfrac{16x^2}{1+\cfrac{9x^2}{1+\cfrac{64x^2}{1+\cfrac{25x^2}{1+\cdots}}}}}}.$$

The coefficients are the squares of the interleaving sequence of odd numbers with multiples of 4

$$1, 4, 3, 8, 5, 12, 7, 16, 9, 20, 11, 24, 13, 28, 15, 32, \ldots,$$

or $b_n = \frac{(n+1)(3-(-1)^n)}{2}$. The sequence b_n/a_n is $1, 4, 1, 4, 1, 4, 1, 4, 1, 4, \ldots$.

We note that

$$1/2=\cfrac{1}{1-\cfrac{1}{1-\cfrac{4}{1-\cfrac{1}{1-\cfrac{4}{1-\cfrac{1}{1-\cdots}}}}}}.$$

We have seen that $\mathrm{cn}(x, 1/4)$ expands to give the sequence

$$1, 0, -1, 0, 2, 0, -13, 0, 161, 0, -3094, 0, 87773, \ldots.$$

The un-aerated sequence

$$1, -1, 2, -13, 161, -3094, 8773, \ldots$$

has generating function

$$\cfrac{1}{1+x-\cfrac{x^2}{1+10x-\cfrac{36x^2}{1+29x-\cfrac{225x^2}{1+58x-\cdots}}}}.$$

Here, the α sequence is $-(n^2+(2n+1))^2$ and the β sequence is $((n+1)(2n+1))^2$.

The un-aerated sequence

$$1, -1, 2, -13, 161, -3094, 8773, \ldots$$

is the moment sequence for the family of orthogonal polynomials

$$P_n(x) = (x + ((n-1)^2 + (2n-1)^2))P_{n-1}(x) - ((n-1)(2n-3))^2 P_{n-2}(x),$$

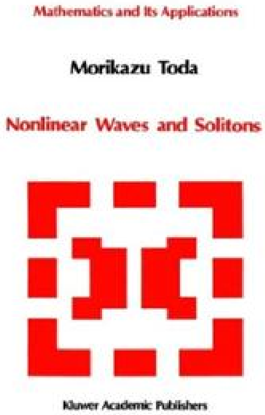
with

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x + 1 \end{aligned}$$

The Hankel transform of this sequence is then given by

$$h_n = \prod_{k=0}^{n} ((k+1)(2k+1))^{2(n-k)}.$$

The Toda chain equations

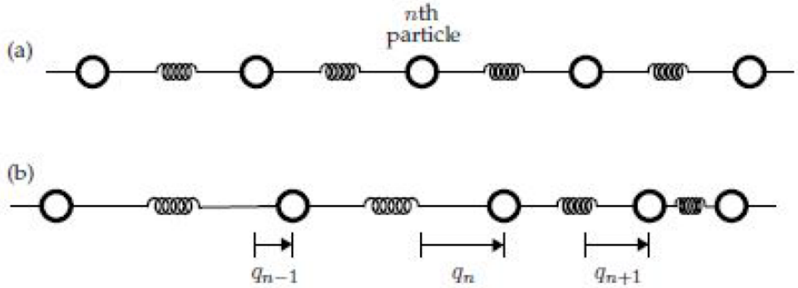

Mathematics and Its Applications
Morikazu Toda
Nonlinear Waves and Solitons
Kluwer Academic Publishers

A little bit of history

- The FPU experiment
- The Korteweg-deVries (KdV) equation
- The Toda chain

The FPU experiment

In the summer of 1953 Fermi, Pasta, Ulam (and Mary Tsingou) conducted numerical experiments on a linear chain of nearest neighbour interactions using non-linear restoring forces. This numerical experiment is often regarded as the birth of non-linear science.



$$m\ddot{q}_j = k(q_{j+1} - 2q_j + q_{j-1})(1 + \alpha(q_{j+1} - q_{j-1})).$$

Korteweg-deVries equation

Korteweg-deVries equation

$$u_t + uu_x + u_{xxx} = 0.$$

For example,

$$u(x,t) = 3v\mathrm{sech}^2 \frac{\sqrt{v}}{2}(x - vt).$$

Integrable systems

The KdV equation has integrals of motion given by

$$\int_{-\infty}^{\infty} P_{2n-1}(u, u_x, u_{xx}, \ldots)\, dx,$$

where

$$P_1 = u, \quad P_n = -\frac{dP_{n-1}}{dx} + \sum_{i=1}^{n-2} P_i P_{n-1-i}.$$

Example. Take

$$u(x,t) = \frac{1}{2}\mathrm{sech}^2\left(\frac{x-t}{2}\right).$$

Then the sequence $\frac{1}{2}\int_{-\infty}^{\infty} P_{2n-1}(x)\, dx$ gives

$$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots.$$

The Toda chain equation

In 1967, Morikazu Toda developed an integrable system inspired by the FPU experiment by using exponential restoring forces. The Toda chain equation is

$$\ddot{y}_n = e^{y_{n+1}-y_n} - e^{y_n - y_{n-1}}.$$

By setting

$$\beta_n = e^{y_{n+1}-y_n}$$

and

$$\alpha_n = \dot{y}_n,$$

we obtain

$$\dot{\beta}_n = e^{y_{n+1}-y_n}(\dot{y}_{n+1} - \dot{y}_n) = \beta_n(\alpha_{n+1} - \alpha_n),$$

and

$$\dot{\alpha}_n = \ddot{y}_n = \beta_n - \beta_{n-1}.$$

Toda

Thus from the equation

$$\ddot{y}_n = e^{y_{n+1}-y_n} - e^{y_n - y_{n-1}}$$

we get the equivalent system

$$\dot{\beta}_n = \beta_n(\alpha_{n+1} - \alpha_n), \quad \dot{\alpha}_n = \beta_n - \beta_{n-1}.$$

By letting

$$y_n = \log \frac{\tau_{n-1}}{\tau_n}, \quad \beta_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \quad \alpha_n = \frac{d}{dt} \log \frac{\tau_{n-1}}{\tau_n},$$

we obtain the bilinear Toda equation

$$\tau_n^{''}\tau_n - (\tau_n^{'})^2 = \tau_{n-1}\tau_{n+1}.$$

Note that

$$\alpha_n = \frac{d}{dt} \log \frac{\tau_{n-1}}{\tau_n} = \frac{\dot{\tau}_{n-1}}{\tau_{n-1}} - \frac{\dot{\tau}_n}{\tau_n}.$$

The Hankel transform 2

We define the Hankel transform of the sequence a_n to be the sequence h_n of Hankel determinants

$$h_0 = |a_0|, h_1 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, h_2 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \ldots,$$

$$h_n = |a_{i+j}|_{0 \le i,j \le n}.$$

Example

For each of the three sequences C_n, C_{n+1} and $C_{n/2}\frac{1+(-1)^n}{2}$, we have

$$h_n \equiv 1.$$

Orthogonal polynomials - revision

Let $P_n(x)$ be a sequence of polynomials that obey a three-term recurrence

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x),$$

with $\beta_0 P_{-1}(x) = 0$ and $P_0(x) = 1$. Then $P_n(x)$ is a family of (monic) orthogonal polynomials. We have

$$\int P_n P_m d\,\mu(x) = \delta_{mn},$$

for an appropriate measure $\mu(x)$. Letting $a_n = \int x^n d\mu(x)$ then, for instance,

$$P_2(x) = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ 1 & x & x^2 \end{vmatrix} / \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} = h_2(x)/h_1$$

tau-function

We have

$$\beta_n = \frac{h_{n-1}h_{n+1}}{h_n^2}$$

and

$$\alpha_n = \frac{h^*_{n+1}}{h_{n+1}} - \frac{h^*_n}{h_n},$$

where for instance

$$h^*_2 = \begin{vmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \end{vmatrix}.$$

Question: when can $\tau_n = h_n$ provide a solution to the Toda chain equations?

Example

We consider the matrix

$$A = [1, e^x - 1] \cdot [e^x, x] = \left[e^{e^x - 1}, e^x - 1\right].$$

Then P_A begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots \\ 0 & 2 & 3 & 1 & \dots \\ 0 & 0 & 3 & 4 & \dots \\ & & \dots & & \dots \end{pmatrix}$$

The inverse matrix

$$A^{-1} = [e^{-x}, \ln(1 + x)]$$

is the coefficient array of the Charlier polynomials.

Toda Example 1

We consider the exponential Riordan array

$$\left[e^{-xe^t}, \ln(1+x)\right].$$

We have

$$\left[e^{-xe^t}, \ln(1+x)\right]^{-1} = \left[e^{e^t(e^x-1)}, e^x-1\right].$$

The production matrix of this inverse has generating function

$$e^{yz}(e^t(1+z)+y(1+z)),$$

corresponding to

$$\alpha_n(t) = n + e^t, \quad \beta_n(t) = ne^t.$$

Then we have

$$\dot{\beta}_n = \beta_n(\alpha_n - \alpha_{n-1}), \quad \dot{\alpha}_n = \beta_{n+1} - \beta_n.$$

We also have

$$\frac{dP_{n+1}(x,t)}{dt} = -(n+1)e^t P_n(x,t) = -\beta_{n+1}P_n(x,t).$$

Moments

We note that the moments $\mu_n = \int x^n d\mu(x)$ for the above orthogonal polynomials are given by

$$\mu_n(t) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} e^{kt},$$

where

$$\left\{ {n \atop k} \right\} = S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

In particular, $\mu_n(0)$ are the Bell numbers.

Toda Example 2

The exponential Riordan array

$$\left[\frac{1}{\sqrt{1-2x\tanh(t)-x^2\mathrm{sech}^2(t)}}, \ln\sqrt{\frac{1+xe^{-t}\mathrm{sech}(t)}{1-xe^{t}\mathrm{sech}(t)}}\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n(t)$ for which

$$\beta_n(t) = -n^2\mathrm{sech}^2(t), \quad \alpha_n(t) = -(2n+1)\tanh(t).$$

Again, we have the Toda equations

$$\dot{\beta}_n = \beta_n(\alpha_n - \alpha_{n-1}),$$

and

$$\dot{\alpha}_n = \beta_{n+1} - \beta_n.$$

We also have

$$\frac{dP_{n+1}(x,t)}{dt} = (n+1)^2 sech(t)^2 P_n(x,t) = -\beta_{n+1}P_n(x,t).$$

Moments

The moments m_n of the previous family of orthogonal polynomials are given by the first column of the inverse array which is

$$\left[\frac{\operatorname{sech}(x+t)}{\operatorname{sech}(t)}, \sinh(t)\frac{\operatorname{sech}(x+t)}{\operatorname{sech}(t)}\right].$$

Thus the moments m_n are given by

$$m_n = n![x^n]\frac{\operatorname{sech}(x+t)}{\operatorname{sech}(t)} = \frac{1}{\operatorname{sech}(t)}\frac{d^n}{dt^n}\operatorname{sech}(t).$$

The Hankel transform of m_n is given by

$$h_n = (-1)^{\binom{n+1}{2}}\operatorname{sech}(t)^{n(n+1)}\prod_{k=0}^{n}(k!)^2.$$

Toda Example 3

The exponential Riordan array

$$\left[e^{-2(z-t)x+x^2}, x\right]$$

is the coefficient array of a family of orthogonal polynomials with

$$\beta_n = -2n, \quad \alpha_n = 2(z-t).$$

We have the Toda equations

$$\dot{\beta}_n = \beta_n(\alpha_n - \alpha_{n-1}),$$

and

$$\dot{\alpha}_n = \beta_{n+1} - \beta_n.$$

We also have

$$\frac{dP_{n+1}(x,t)}{dt} = 2(n+1)P_n(x,t) = \beta_{n+1}P_n(x,t).$$

Moments

The moments m_n of the previous family of orthogonal polynomials are given by the first column of

$$\left[e^{-2(z-t)x+x^2}, x\right]^{-1} = \left[e^{2(z-t)x-x^2}, x\right].$$

We obtain

$$m_n = H_n(z-t)$$

where $H_n(x)$ is the n-th Hermite polynomial. The Hankel transform of m_n is given by

$$h_n = (-2)^{\binom{n+1}{2}} \prod_{k=0}^{n} k!$$

Generalized Toda Example 1

The exponential Riordan array

$$\left[\frac{1}{(1+tx)}, \ln\left(\frac{1+(t+1)x}{1+tx}\right)\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n(x,t)$ whose coefficients are given by

$$\alpha_n = n + (2n+1)t, \quad \beta_n = n^2 t(t+1).$$

Then α_n and β_n satisfy the modified Toda equations

$$\dot{\beta}_n = \frac{1}{t(t+1)}\beta_n(\alpha_n - \alpha_{n-1}),$$

and

$$\dot{\alpha}_n = \frac{1}{t(t+1)}(\beta_{n+1} - \beta_n).$$

We also have

$$\frac{dP_{n+1}(x,t)}{dt} = -(n+1)^2 P_n(x,t) = -\frac{\beta_{n+1}}{t(t+1)} P_n(x,t).$$

Generalized Toda Example 2

The exponential Riordan array

$$\left[\frac{1}{1+tx}, \frac{x}{1+tx}\right]$$

is the coefficient array of the family of orthogonal polynomials whose coefficients are given by

$$\alpha_n = (2n+1)t, \quad \beta_n = n^2 t^2.$$

Then α_n and β_n satisfy the modified Toda equations

$$\dot{\beta}_n = \frac{1}{t^2}\beta_n(\alpha_n - \alpha_{n-1}),$$

and

$$\dot{\alpha}_n = \frac{1}{t^2}(\beta_{n+1} - \beta_n).$$

We also have

$$\frac{dP_{n+1}(x,t)}{dt} = -(n+1)^2 P_n(x,t) = -\frac{\beta_{n+1}}{t^2} P_n(x,t).$$

Integer sequences

We can describe integer sequences in a number of ways. Two common ways are

- Generating function
- Recurrence

Example

- $$r^n = [x^n]\frac{1}{1-rx}$$
- $$F_n = [x^n]\frac{x}{1-x-x^2}$$
- $$F_n = F_{n-1} + F_{n-2},$$ $$\text{with} \quad F_0 = 0, F_1 = 1$$

Integer sequences

Sometimes, the recurrence may be more involved.

Example

$$a_n = a_{n-1} + \sum_{i=0}^{n-3} a_i a_{n-1-i}$$

with

$$a_0 = 0, a_1 = 2, a_2 = 1.$$

This gives us the sequence

$$0, 2, 1, 1, 3, 6, 14, 33, 79, 194, \ldots.$$

Somos sequences

Somos 4.

$$a_n = \frac{\alpha a_{n-1} a_{n-3} + \beta a_{n-2}^2}{a_{n-4}}, \quad n \geq 4.$$

Somos 5.

$$a_n = \frac{\alpha a_{n-1} a_{n-4} + \beta a_{n-2} a_{n-3}}{a_{n-5}}, \quad n \geq 5.$$

Somos 6.

$$a_n = \frac{\alpha a_{n-1} a_{n-5} + \beta a_{n-2} a_{n-4} + \gamma a_{n-3}^2}{a_{n-6}}, \quad n \geq 6.$$

Binomial transform

If $a_n = [x^n]g(x)$, then the sequence

$$b_n = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} b_k,$$

where

$$b_n = [x^n]\frac{1}{1-rx}g\left(\frac{x}{1-rx}\right) = \left(\frac{1}{1-rx}, \frac{x}{1-rx}\right) \cdot g(x)$$

is called the r-th binomial transform of a_n. The inverse binomial transform corresponds to $r = -1$.

INVERT transform

If

$$a_n = [x^n]g(x),$$

then the r-th INVERT transform of the sequence a_n has g.f. given by

$$\frac{g(x)}{1 - rxg(x)} = (g(x), xg(x)) \cdot \frac{1}{1 - rx}.$$

The r-th inverse INVERT transform has g.f. given by

$$\frac{g(x)}{1 + rxg(x)}.$$

Reversion of a sequence

If $f(x)$ is a power series with $f(0) = 0$, then the reversion of f, denoted by

$$\bar{f}(x) = \mathrm{Rev}\{f\}(x)$$

is the solution u of

$$f(u) = x$$

such that

$$u(0) = 0.$$

If $a_n = [x^n]g(x)$ is sequence $a_0 \neq 0$, we shall call *reversion of* a_n the sequence b_n such that

$$b_n = [x^n]\frac{1}{x}\mathrm{Rev}\{xg(x)\}.$$

Example The reversion of the binomial transform of a sequence a_n is the inverse INVERT transform of the reversion of a_n.

Reversion of a sequence

If $a_n = [x^n]g(x)$ then its reversion b_n is the first column of

$$(g(x), xg(x))^{-1}$$

We have

$$\begin{aligned}
&\left(\frac{1}{1-x}g\left(\frac{x}{1-x}\right), \frac{x}{1-x}g\left(\frac{x}{1-x}\right)\right)^{-1} \\
&= \left(\left(\frac{1}{1-x}, \frac{x}{1-x}\right)\cdot(g(x), xg(x))\right)^{-1} \\
&= (g(x), xg(x))^{-1}\cdot\left(\frac{1}{1+x}, \frac{x}{1+x}\right) \\
&= \left(\frac{1}{x}\mathrm{Rev}\{xg(x)\}, \mathrm{Rev}\{xg(x)\}\right)\cdot\left(\frac{1}{1+x}, \frac{x}{1+x}\right) \\
&= \left(\frac{1}{x}\mathrm{Rev}\{xg(x)\}\frac{1}{1+x\frac{1}{x}\mathrm{Rev}\{xg(x)\}}, \ldots\right).
\end{aligned}$$

Quadratic equations

$$au^2 + bu + c = 0$$
$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example

$$u(1-u) = x \quad \text{or} \quad u^2 - u + x = 0$$

The solution is

$$u = \frac{1 \pm \sqrt{1-4x}}{2}.$$

We ask that $u(0) = 0$. This gives us

$$\mathrm{Rev}\{x(1-x)\} = \frac{1 - \sqrt{1-4x}}{2}.$$

Quadratic equations

The Taylor series expansion of $\frac{1-\sqrt{1-4x}}{2}$ at 0 is

$$x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + \cdots .$$

The numbers C_n given by the non-zero coefficients

$$1, 1, 2, 5, 14, 42, 429, \ldots$$

are the Catalan numbers. We have

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

We write

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + \cdots$$

Quadratic equations

We have

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Then

$$au^2 + bu + c = 0$$

has solution

$$u = -\frac{c}{b} C\left(\frac{ac}{b^2}\right).$$

Cubic equations

Trigonometric approach. Starting with

$$ax^3 + bx^2 + cx + d = 0$$

use the substitution

$$x = t - \frac{b}{3a}$$

to get the depressed cubic equation

$$t^3 + pt + q = 0.$$

Set

$$t = u \cos \theta$$

and compare with the identity

$$4 \cos^3 \theta - 3 \cos \theta - \cos(3\theta) = 0$$

Cubic equations

Consider the equation

$$u(1-u^2)=x \quad \text{or} \quad u^3-u+x=0$$

We find that

$$u=\frac{2}{\sqrt{3}}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{\sqrt{27}x}{2}\right)\right)$$

is the solution with $u(0)=0$. This expression is the generating function for the integer sequence

$$0,1,0,1,0,3,0,12,0,55,0,273,0,1428,0,\ldots$$

where the numbers t_n that begin

$$1,1,3,12,55,\ldots$$

are the ternary numbers

$$t_n=\frac{1}{2n+1}\binom{3n}{n}.$$

The Hankel transform

We define the Hankel transform of the sequence a_n to be the sequence h_n of Hankel determinants

$$h_0 = |a_0|, h_1 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, h_2 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \ldots,$$

$$h_n = |a_{i+j}|_{0 \le i,j \le n}.$$

Example

Each of the three sequences

$$1, 1, 2, 5, 14, 42, \ldots,$$

$$1, 2, 5, 14, 42, 429, \ldots,$$

$$1, 0, 1, 0, 2, 0, 5, 0, 14, 0, \ldots,$$

has Hankel transform

$$1, 1, 1, 1, \ldots.$$

Hankel transform of the ternary numbers

The Hankel transforms of the sequences

$$1, 1, 3, 12, 55, 273, \ldots,$$

$$1, 3, 12, 55, 273, \ldots,$$

$$1, 0, 1, 0, 3, 0, 12, 0, 55, 0, 273, \ldots$$

are given by the sequences

$$1, 2, 11, 170, 7429, 920460, \ldots,$$

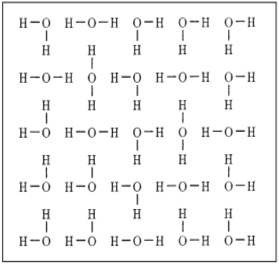
respectively

$$1, 3, 26, 646, 45885, \ldots,$$

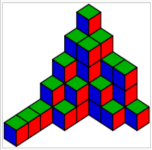
respectively

$$1, 1, 2, 6, 33, 286, 4420, 109820, 4799134, \ldots.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



square ice



Hankel transform

A classical result says that

$$\beta_n = \frac{h_{n-1} h_{n+1}}{h_n^2}$$

and

$$\alpha_n = \frac{h'_n}{h_n} - \frac{h'_{n-1}}{h_{n-1}} + 0^n,$$

where, for example,

$$h'_2 = \begin{vmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \end{vmatrix}$$

Hankel transform

Let h_n be the Hankel transform of a sequence $a_n = [x^n]g(x)$. Then h_n is also the Hankel transform of

- $(-1)^n a_n$
- the r-th binomial transform $\sum_{k=0}^{n} \binom{n}{k} r^{n-k} a_k$,
- the r-th INVERT transform of a_n, with g.f. $\frac{g(x)}{1-rxg(x)}$.

Let a_n have Hankel transform

$$h_0, h_1, h_2, \ldots.$$

Then the sequence b_n with

$$b_n = [x^n]\frac{1}{1-x-x^2g(x)}$$

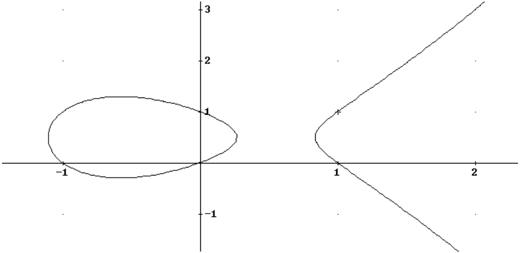
has Hankel transform

$$1, h_0, h_1, h_2, \ldots$$

Elliptic curves

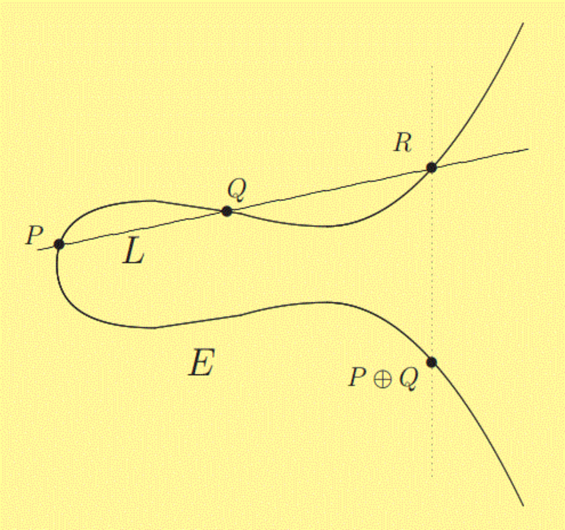
An elliptic curve can be defined by any of the equations

- $y^2 = x^3 + Ax + B$
- $y^2 = 4x^3 - g_2x - g_3$
- $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$



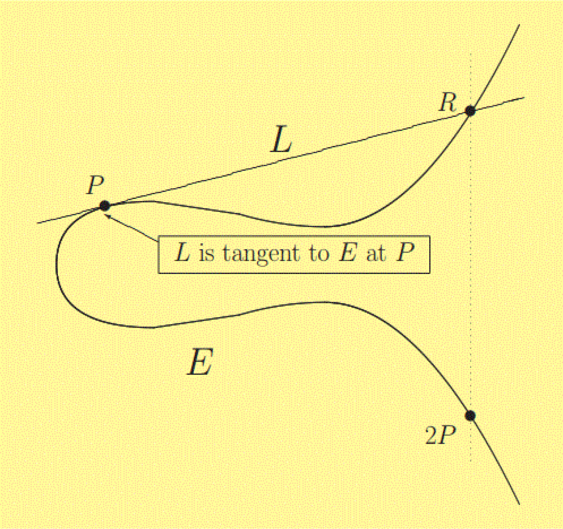
Elliptic curves have a group structure: points can be added. The point at infinity is the identity.

Adding points on an elliptic curve



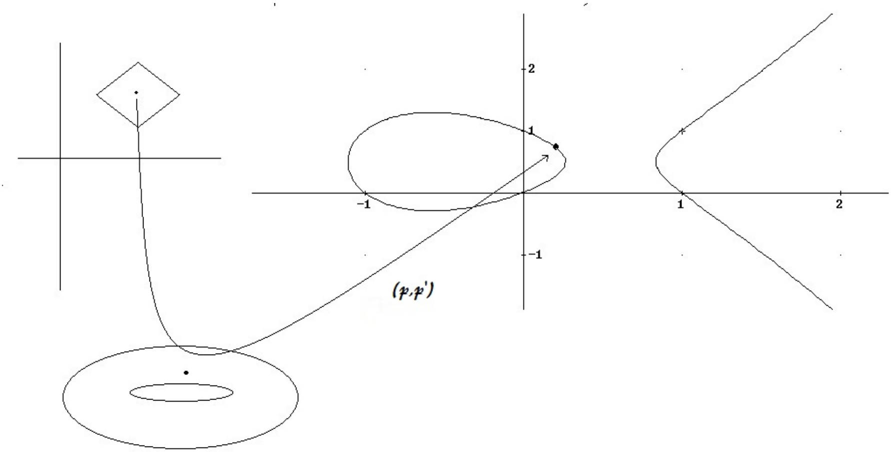
(after Silverman)

Adding a point to itself



(after Silverman)

Elliptic curve: parametrisation



Weierstrass $\wp$ function

- $$\wp(z;\omega_1,\omega_2)=\frac{1}{z^2}+\sum_{0\neq\omega\in\mathbb{Z}\omega_1+\mathbb{Z}\omega_2}\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}.$$
- $$\wp'^2=4\wp^3-g_2\wp-g_3$$
- $(\wp,\wp')$ provides a parametrisation for the curve $y^2=4x^3-g_2x-g_3$.
- $$\sigma(z)=z\prod_{0\neq\omega\in\Omega}\left(1-\frac{z}{\omega}\right)\cdot e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega^2}\right)}$$
- $$\zeta(z)=\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{z}+\sum_{0\neq\omega\in\Omega}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right)$$
- $$\zeta'(z)=-\wp\Rightarrow\wp=-\frac{d^2}{dz^2}\ln\sigma$$

Weierstrass σ and division polynomials ψ_n

If $P=(0,0)=(\wp(z),\wp'(z))$ is a point on the elliptic curve E, then

$$(nP)_x=-\frac{\psi_{n-1}(z)\psi_{n+1}(z)}{\psi_n(z)^2}$$

and

$$(nP)_y=\frac{\psi_{2n}}{2\psi_n^4}$$

where

$$\psi_n(z)=\frac{\sigma(nz)}{\sigma(z)^{n^2}}.$$

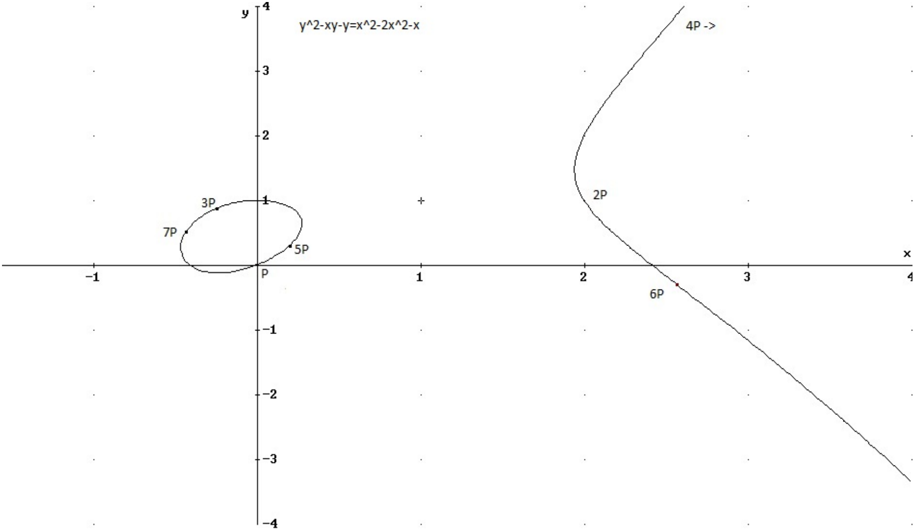
We have the **Kiepert formula (1873)**

$$\psi_n = \frac{\sigma(nu)}{\sigma(u)^{n^2}} = \frac{1}{(-1)^{n-1}(1!2!\cdots(n-1)!)^2} \begin{vmatrix} \wp'(u) & \wp''(u) & \cdots & \wp^{(n-1)}(u) \\ \wp''(u) & \wp'''(u) & \cdots & \wp^{(n)}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)}(u) & \wp^{(n)}(u) & \cdots & \wp^{(2n-3)}(u) \end{vmatrix}$$

Swart formula (2003)

$$s_n = (-1)^{\frac{n(n+1)}{2}}(x_{n-1}-\bar{x})(x_{n-2}-\bar{x})^2\cdots(x_1-\bar{x})^n s_0\left(\frac{s_{-1}}{s_0}\right)^n.$$

Elliptic curve: nP



Sequences from elliptic curves

We consider the equation

$$y^2 - 3xy - y = x^3 - x$$

Solving for y, we find that

$$y = \frac{1 + 3x + \sqrt{1 + 2x + 9x^2 + 4x^2}}{2}.$$

This expands to give the sequence

$$1, 2, 2, -1, -3, 7, 4, -38, 27, 175, \ldots.$$

We shed the first two terms to arrive at

$$2, -1, -3, 7, 4, -38, 27, 175, -384, -546, \ldots,$$

with g.f. of

$$g(x) = \frac{\sqrt{1 + 2x + 9x^2 + 4x^3} - x - 1}{2x^2} = \left(\frac{2+x}{1+x}, \frac{-x^2(2+x)}{(1+x)^2}\right) \cdot C(x).$$

Sequences from elliptic curves

This sequence has general term

$$a_n = \sum_{k=0}^{n} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n-j}{n-2k-j} (-1)^{n-k-j} 2^{k+1-j} C_k.$$

The sequence

$$2, -1, -3, 7, 4, -38, 27, 175, -384, -546, \ldots$$

has a Hankel transform that begins

$$2, -7, -57, 670, 23647, -833503, \ldots,$$

which is a $(1, 16)$ Somos 4 sequence.

The shifted sequence

$$-1, -3, 7, 4, -38, 27, 175, -384, -546, \ldots$$

has generating function given by

$$-\left(\frac{1+4x}{1+x+4x^2}, \frac{x^3(1+4x)}{(1+x+4x^2)^2}\right) \cdot C(x).$$

It has a Hankel transform that begins

$$-1, -16, 113, 3983, -140576, -14871471, \ldots.$$

This is also a $(1, 16)$ Somos 4 sequence.

We now form the generating function

$$\frac{1}{1-x+x^2g(x)}=\frac{2}{1-3x+\sqrt{1+2x+9x^2+4x^3}}$$

which expands to give the sequence

$$1, 1, -1, -2, 4, 3, -21, 12, 98, -198, -322, \ldots$$

with Hankel transform

$$1, -2, -7, 57, 670, -23647, \ldots.$$

We can express the generating function as

$$\left(\frac{1}{1-3x}, -\frac{x(2+x^2)}{(1-3x)^2}\right)\cdot C(x).$$

We revert this last sequence to get the sequence with g.f.

$$\frac{1+3x-\sqrt{1+6x+9x^2-4x^3-8x^4}}{2x^3} = \frac{1+2x}{1+3x} C\left(\frac{x^3(1+2x)}{(1+3x)^2}\right).$$

This sequence begins

$$1, -1, 3, -8, 22, -59, 155, -396, 978, -2310, 5122, \ldots.$$

Its Hankel transform is given by

$$1, {\color{red}2}, 1, {\color{red}-7}, -16, {\color{red}-57}, -113, {\color{red}670}, 3983, {\color{red}23647}, 140576, \ldots,$$

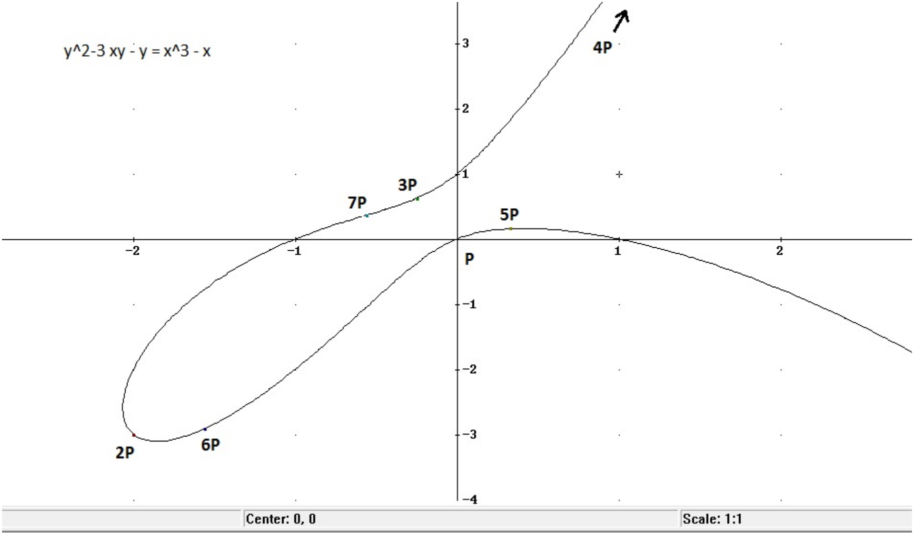
which is a $(1, -2)$ Somos 4 sequence, which coincides with the elliptic divisibility sequence of the curve.

Elliptic divisibility sequence

Thus we have recovered the elliptic divisibility sequence of the elliptic curve

$$y^2 - 3xy - y = x^3 - x.$$

E =ellinit$([-3, 0, -1, -1, 0])$;
$P = [0, 0]$;
z=ellpointtoz(E,P);
al =List();
for($i = 1, 10$,listput(al,round(ellsigma(E,$i * z$)/ellsigma(E,z)$^{i^2}$)));al
$[1, 1, 2, 1, -7, -16, -57, -113, 670, 3983]$
or
for($i = 1, 10$, listput(al,subst(elldivpol(E,i),x,0)));al
$[1, 1, 2, 1, -7, -16, -57, -113, 670, 3983]$


y^2-3 xy - y = x^3 - x
4P
3P
7P
5P
P
2P
6P
3
2
1
−1
−2
−3
−4
−2
−1
1
2
Center: 0, 0
Scale: 1:1

Coordinates of nP on $y^2 - 3xy - y = x^3 - x$

We have the following (x, y) coordinates for nP on the curve $y^2 - 3xy - y = x^3 - x$, where $P = [0, 0]$.

$(nP)_x$	0	-2	$-\frac{1}{4}$	14	$\frac{16}{49}$	$\frac{-399}{256}$	$\frac{-1808}{3249}$
$(nP)_y$	0	-3	$\frac{5}{8}$	78	$\frac{55}{343}$	$\frac{-11921}{4096}$	$\frac{68464}{185193}$
$\frac{y}{x}$	1	$\frac{3}{2}$	$-\frac{5}{2}$	$\frac{39}{7}$	$\frac{55}{122}$	$\frac{703}{912}$	$-\frac{4279}{6441}$

We form the continued fraction

$$\cfrac{1}{1+x-\cfrac{2x^2}{1+\frac{3x}{2}-\cfrac{\frac{x^2}{4}}{1-\frac{5x}{2}+\cfrac{14x^2}{1+\cfrac{39x}{7}+\cfrac{\frac{16x^2}{49}}{1+\frac{55x}{112}-\cfrac{\frac{399x^2}{256}}{1-\dots}}}}}}.$$

The continued fraction

$$\cfrac{1}{1+x-\cfrac{2x^2}{1+\frac{3x}{2}-\cfrac{\frac{x^2}{4}}{1-\frac{5x}{2}+\cfrac{14x^2}{1+\cfrac{39x}{7}+\cfrac{\frac{16x^2}{49}}{1+\frac{55x}{112}-\cfrac{\frac{399x^2}{256}}{1-\ldots}}}}}}$$

expands to give the sequence

$$1, -1, 3, -8, 22, -59, 155, -396, 978, -2310, 5122, \ldots$$

with g.f

$$\left(\frac{1+2x}{1+3x}, \frac{x^3(1+2x)}{(1+3x)^2}\right) \cdot C(x).$$

We have

$$a_n = \sum_{k=0}^{n} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n-k-j}{n-3k-j} 2^j (-3)^{n-k-j} C_k.$$

We note that the second binomial transform of the sequence

$$1, -1, 3, -8, 22, -59, 155, -396, 978, -2310, 5122, \ldots$$

is the sequence with g.f.

$$\left(\frac{1}{(1+x)(1-2x)}, \frac{x^3}{(1+x)^2(1-2x)^2}\right) \cdot C(x).$$

This is the sequence

$$1, 1, 3, 6, 14, 33, 79, 194, 482, 1214, 3090, \ldots.$$

Now recall that the recurrence

$$a_n = a_{n-1} + \sum_{i=0}^{n-3} a_i a_{n-1-i}$$

with

$$a_0 = 0, a_1 = 2, a_2 = 1,$$

has solution

$$\color{red}{0}, \color{red}{2}, 1, 1, 3, 6, 14, 33, 79, 194, \ldots.$$

Elliptic curves and Riordan arrays

We have seen that the elliptic curve

$$E : y^2 - 3xy - y = x^3 - x$$

gives rise to the following Riordan arrays.

- $\left(\frac{2+x}{1+x}, -\frac{x^2(2+x)}{(1+x)^2}\right)$
- $\left(\frac{1+4x}{1+x+4x^2}, \frac{x^3(1+4x)}{(1+x+4x^2)^2}\right)$
- $\left(\frac{1}{1-3x}, \frac{x(2+x^2)}{(1-3x)^2}\right)$
- $\left(\frac{1+2x}{1+3x}, \frac{x^3(1+2x)}{(1+3x)^2}\right)$
- $\left(\frac{1}{1-x-2x^2}, \frac{x^3}{(1-x-2x^2)^2}\right)$.

$y^2 - xy - y = x^3 - x^2 - x$ and Motzkin paths

We solve

$$y^2 - xy - y = x^3 - x^2 - x$$

to get the the sequence

$$\color{red}{0}, \color{red}{1}, 1, 0, 1, 1, 3, 5, 12, 24, 55, 119, 272, \ldots$$

with g.f. $\frac{1+x-\sqrt{1-2x-3x^2+4x^3}}{2}$. The sequence

$$1, 0, 1, 1, 3, 5, 12, 24, 55, 119, 272, \ldots$$

($A090345$, Motzkin paths with no level steps at even levels) has g.f.

$$g(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2 + 4x^3}}{2x^2} = C\left(\frac{x^2}{1-x}\right).$$

We have

$$g(x) = \cfrac{1}{1 - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - \cdots}}}}}.$$

$y^2 - xy - y = x^3 - x^2 - x$ and Motzkin paths

We form the g.f.

$$\frac{1}{1-x-x^2g(x)} = \frac{1-x-\sqrt{1-2x-3x^2+4x^3}}{2x^2(1-x)},$$

which expands to give the sequence

$$1, 1, 2, 3, 6, 11, 23, 47, 102, 221, 493, \ldots$$

or the number of Motzkin paths with no level steps at odd level ($A090344$). We have

$$g(x) = \cfrac{1}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-x\cfrac{x^2}{1-\cfrac{x^2}{1-\cdots}}}}}.$$

$y^2 - xy - y = x^3 - x^2 - x$ and Narayana numbers

We now revert this last sequence to get the sequence ($A129509$)

$$1, -1, 0, 2, -4, 3, 5, -20, 29, -1, -94, \ldots$$

with g.f.

$$\frac{1 + x + x^2 - \sqrt{1 + 2x + 3x^2 - 2x^3 + x^4}}{2x^3} = \frac{1}{1 + x + x^2} C\left(\frac{x^3}{(1 + x + x^2)^2}\right).$$

These numbers are the diagonal sums of the signed Narayana triangle

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & -6 & -6 & -1 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 \\ -1 & -15 & -50 & -50 & -15 & -1 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 \end{array}\right)$$

$y^2 - xy - y = x^3 - 2x^2 - x$ and generalized Narayana numbers

In a similar fashion, for the curve

$$y^2 - xy - y = x^3 - 2x^2 - x$$

we get the sequence

$$1, -1, -1, 4, -4, -5, 23, -28, -28, 164, -232, \ldots$$

with g.f.

$$\frac{1 + x + 2x^2 - \sqrt{1 + 2x + 5x^2 + 4x^3}}{2x^3} = \left(\frac{1}{1 + x + 2x^2}, \frac{x^3}{(1 + x + 2x^2)^2}\right) \cdot C(x).$$

This is given by the diagonal sums of the generalized Narayana triangle

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 4 & 0 & 0 & 0 & 0 \\ -1 & -9 & -18 & -8 & 0 & 0 & 0 \\ 1 & 14 & 50 & 56 & 16 & 0 & 0 \\ -1 & -20 & -110 & -220 & -160 & -32 & 0 \\ 1 & 27 & 210 & 645 & 840 & 432 & 64 \end{array}\right)$$

The last triangle has bivariate g.f. given by

$$\cfrac{1}{1+x+2xy-\cfrac{x^2y}{1+x+2xy-\cfrac{x^2y}{1-\cdots}}}$$

and so the sequence has g.f. given by

$$\cfrac{1}{1+x+2x^2-\cfrac{x^3}{1+x+2x^2-\cfrac{x^3}{1-\cdots}}}.$$

Using the coordinates of $n[0,0]$ on the curve, this is equivalent to

$$\cfrac{1}{1+x+\cfrac{2x^2}{1+\frac{x}{2}-\cfrac{\frac{x^2}{4}}{1-\frac{7x}{2}+\cfrac{18x^2}{1+\frac{41x}{9}+\cfrac{\frac{16x^2}{81}}{1-\cdots}}}}}$$


3
2
1
−1
−2
−3
−4
−2
−1
1
2
Center: 0, 0
Scale: 1:1

The family of elliptic curves

$$y^2 - rxy - y = x^3 - rx^2 - x$$

gives rise to the integer sequences with g.f.

$$\left(\frac{1-(r+1)x}{1-rx-rx^2}, \frac{x^3(1-(r+1)x)}{(1-rx-rx^2)^2}\right) \cdot C(x).$$

The Hankel transform of these sequences is the elliptic divisibility sequence of the corresponding curve. Taking the $(r+1)$-st binomial transform followed by an inverse $(r+2)$-nd INVERT transform of this sequence we obtain the sequence with g.f. given by

$$\left(\frac{1}{1+(r+2)x+x^2}, \frac{x(2+r+(r+1)x^2)}{(1+(r+2)x+x^2)^2}\right) \cdot C(x).$$

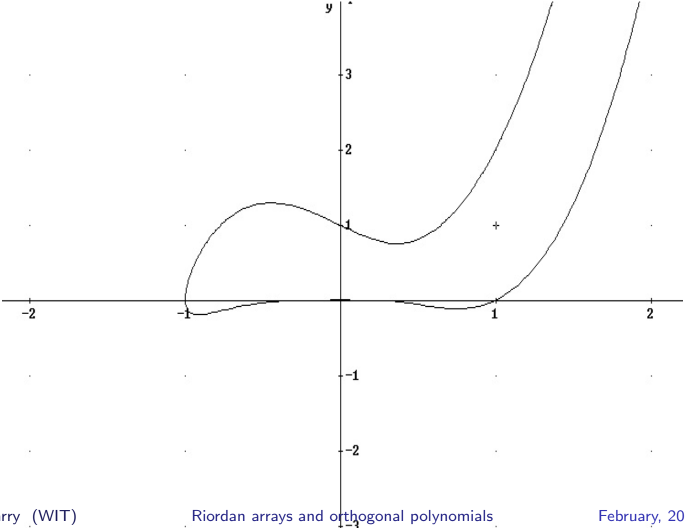
This latter sequence is given by the diagonal sums of $\frac{\binom{n}{k}}{n-k+1}$ times the Riordan array

$$\left(1, -\frac{x(1-(r+1)x)}{1-(r+2)x}\right).$$

Somos 6 and ASMs

We consider the hyper-elliptic curve

$$y^2 - (1 - x + 2x^3)y - x^4(1 - x^2) = 0.$$



Somos 6 and ASMs

Solving for y gives the generating function for the sequence $\sum_{k=0}^{\lfloor \frac{n+1}{3} \rfloor} \binom{n-k+1}{2k}(-1)^k C_k$ which begins

$$1, 1, 0, -2, -5, -7, -4, 10, 38, 70, 68, \ldots$$

Its generating function is given by

$$\left(\frac{1-x^2}{1-x+2x^3}, \frac{x^4(1-x^2)}{(1-x+2x^3)^2}\right) \cdot C(x).$$

Its Hankel transform is

$$1, -1, 1, 2, -2, 1, 3, -3, 1, 4, -4, 1, 5, -5 \ldots$$

which is a simple Somos 6 sequence

$$e_n = \frac{-e_{n-1}e_{n-5} + e_{n-2}e_{n-4} + e_{n-3}^2}{e_{n-6}}.$$

Somos 6 and ASMs

The reversion of the sequence

$$1, 1, 0, -2, -5, -7, -4, 10, 38, 70, 68, \ldots$$

begins

$$1, -1, 2, -3, 7, -14, 36, -85, 228, -587, 1612, -4354, 12166, \ldots$$

Its generating function is

$$\frac{2}{\sqrt{3}} \frac{\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{\sqrt{27}x}{2\sqrt{1+x}}\right)\right)}{\sqrt{1+x}}.$$

Its Hankel transform is given by

$$1, 1, 2, 6, 33, 286, \ldots.$$

This is the number of alternating sign $(2n+1) \times (2n+1)$ matrices symmetric with respect to both horizontal and vertical axes.

Somos 6 and ASMs

In fact, the reversion sequence

$$1, -1, 2, -3, 7, -14, 36, -85, 228, -587, 1612, -4354, 12166, \ldots$$

is given by

$$(-1)^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} t_k,$$

which has the same Hankel transform as the aeration

$$1, 0, 1, 0, 3, 0, 12, 0, 55, 0, 273, \ldots$$

of the ternary numbers t_n. [Trivially, the aerated ternary numbers are the reversion of the solution to

$$y^2 = x^6 - 2x^4 + x^2$$

]

Somos 6 and special Riordan arrays

We finish by noting that the sequences with g.f.

$$\left(\frac{1}{1-rx-x^2-rx^3}, \frac{x^4}{(1-rx-x^2-rx^3)^2}\right)\cdot C(x)$$

have Hankel transforms that are $(1, 1-r^2, r^2-1)$ Somos 6 sequences.

In conclusion, we see that the Hankel transforms of the inversions of solutions of elliptic and hyper-elliptic curve equations can count combinatorially significant objects. These links deserve further study.

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