Enumeration of Young tableaux in a diagonal strip using operator approach

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The Goal

To enumerate standard Young tableaux of certain skew shapes

Machinery

A transfer operator approach, used by Elkies

The Ultimate Goal

To obtain q-analogues

Outline of the talk

① Definition of standard Young tableaux
(or, why do we care about them?)

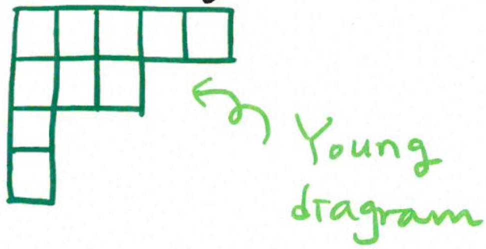
② The operator method used by Elkies

③ Extension and application to more complicated shapes

④ Speculation on extending to q-analogues

$\boxed{1}$ Definition of Standard Young tableaux

Def. · A partition λ is a sequence of non-increasing positive integers.

Ex) $\lambda = (5,3,1,1) =$ 

Notation. · $|\lambda| = \sum_i \lambda_i$; size of λ

· $\lambda \vdash |\lambda|$

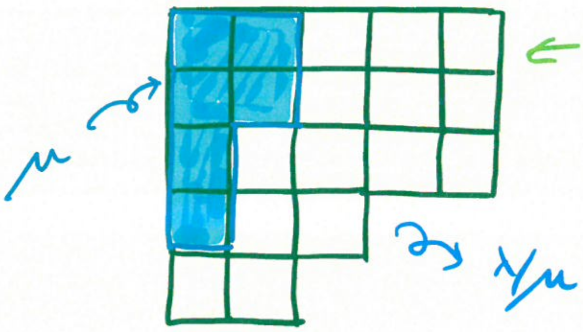
· $\ell(\lambda) = \#$ of parts in λ

; length of λ

Def. A skew Young diagram is the difference $\lambda \setminus \mu$ of two Young diagrams where $\mu \subset \lambda$.

Ex) $\lambda = (5,5,5,3,2)$,

$\mu = (2,2,1,1,0)$



Def A standard Young tableau of shape λ is an array $T = (T_{ij})$ of $1, 2, \ldots, |\lambda|$ of shape λ that is strictly increasing in every column and in every row.

Ex). $\lambda = (4, 3, 1, 1)$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 9 \\ \hline 3 & 5 & 7 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} 8 & & & \\ \cline{1-1}\end{array}$$

- $|T| = \sum T_{ij}$; size of T

Def $f^\lambda = \#$ of SYT of shape λ

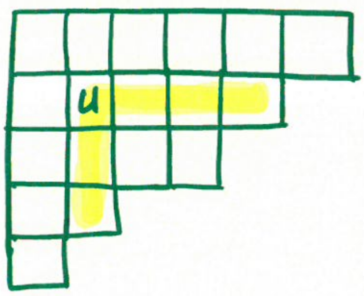
Thm (the hook-length formula)

For $\lambda \vdash n$,

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where $h(u) = \lambda_i + \lambda'_j - i - j + 1$ for $u = (i, j)$.

"hook length"



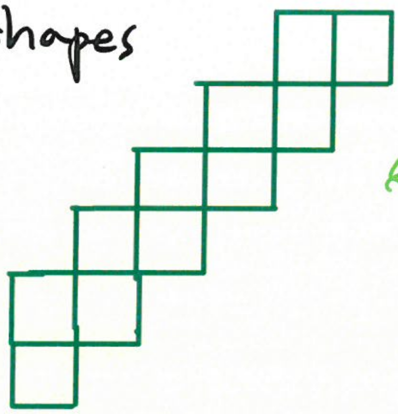
Our interest:

To enumerate SYT of certain skew shapes

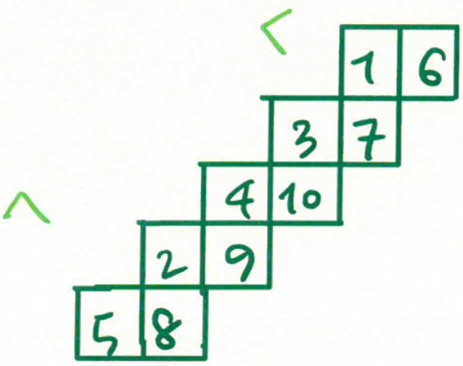
Special case 1

skew staircase shapes,

or "zig-zag" shapes



← "2-strip"



SYT of 2-strip $\longleftrightarrow$ alternating permutations

$$\sigma = (5 < 8 > 2 < 9 > 4 < 10 > 3 < 7 > 1 < 6)$$

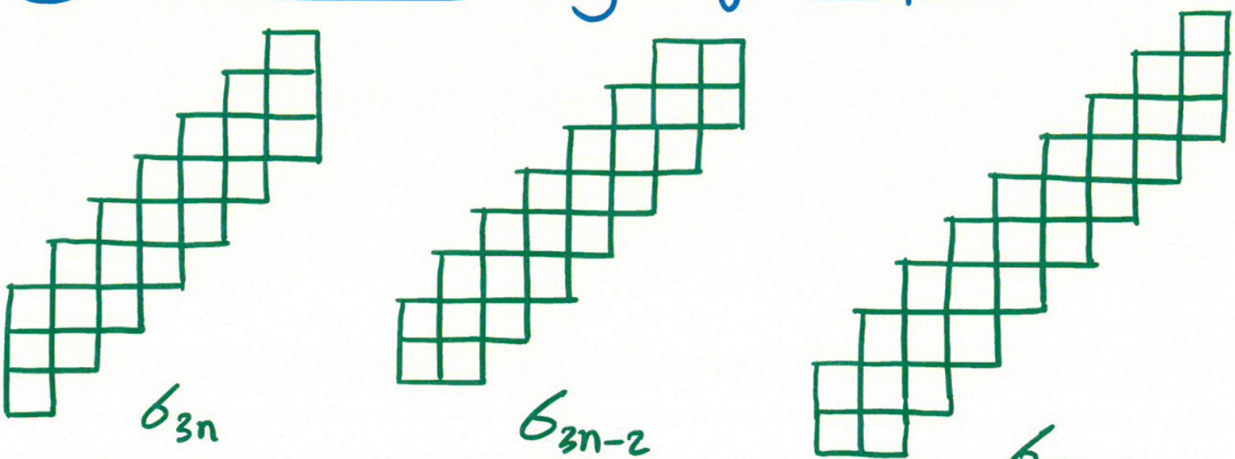
D. André (1879)

- $\mathrm{Alt}_n :=$ the set of alternating permutations of size n
- $E_n = |\mathrm{Alt}_n|$; Euler number

Then the exponential generating function is given by

$$\sum_{n=0}^{\infty} \frac{E_n x^n}{n!} = \sec x + \tan x .$$

② Thickened zig-zag shapes



σ_{3n} σ_{3n-2} σ_{3n-1}

Baryshnikov - Romik ('2010)

$$f^{\sigma_{3n-2}} = \frac{(3n-2)!\, E_{2n-1}}{(2n-1)!\, 2^{2n-2}}$$

$$f^{\sigma_{3n-1}} = \frac{(3n-1)!\, E_{2n-1}}{(2n-1)!\, 2^{2n-1}}$$

$$f^{\sigma_{3n}} = \frac{(3n)!\, (2^{2n-1}-1) E_{2n-1}}{(2n-1)!\, 2^{2n-1} (2^{2n}-1)}$$

$\boxed{2}$ The operator method used by Elkies

"On the Sums $\sum_{k=-\infty}^{\infty} (4k+1)^{-n}$" – Noam D. Elkies

American Mathematical Monthly 110 (2003)

Euler : The sum

$$S(n)=\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} = \begin{cases} (1-2^{-n})\zeta(n), & n \text{ even} \\ L(n,\chi_4), & n \text{ odd} \end{cases},$$

where

$$\zeta(n)=\sum_{k=1}^{\infty} \frac{1}{k^n} = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots,$$

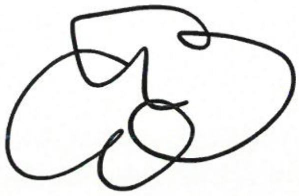
$$L(n,\chi_4)=\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} = 1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\cdots,$$

is a rational multiple of π^n.

??? Why all of sudden ???

Consider the generating function

$$G(z) := \sum_{n=1}^{\infty} S(n) z^n = \sum_{n=1}^{\infty} \Big(\underbrace{\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n}}_{\substack{\text{sum is taken in the order} \\ k=0,-1,1,-2,2,-3,3,\dots}} \Big) z^n ,$$

$\Rightarrow$  very complicated computation

$\Rightarrow$ We find that

$$G(z) = \frac{\pi z}{4} \left(\sec \frac{\pi z}{2} + \tan \frac{\pi z}{2} \right).$$

Calabi ('1993) : n=2 case

$S(2) = \pi^2/8$ via change of variable method

$\boxed{n=2}$ $S(2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

① Write each term $\frac{1}{(2k+1)^2}$ as

$$\int_0^1 \int_0^1 (xy)^{2k} \, dx \, dy$$

and rewrite the sum as

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{k=0}^{\infty} \int_0^1 \int_0^1 (xy)^{2k} \, dx \, dy$$

$$= \int_0^1 \int_0^1 \left(\sum_{k=0}^{\infty} (xy)^{2k} \right) dx \, dy = \int_0^1 \int_0^1 \frac{dx \, dy}{1-(xy)^2}.$$

② Change of variables

$$x = \frac{\sin u}{\cos v}, \quad y = \frac{\sin v}{\cos u}.$$

Note. The Jacobian

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos^2 v} \\ \frac{\sin u \sin v}{\cos^2 u} & \frac{\cos v}{\cos u} \end{vmatrix}$$

$$= 1 - \frac{\sin^2 u \cdot \sin^2 v}{\cos^2 u \cdot \cos^2 v}$$

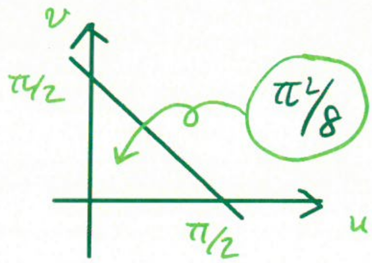
$$= 1 - (xy)^2.$$

Thus, $\displaystyle\int_0^1 \int_0^1 \frac{dx\,dy}{1-(xy)^2} = \iint_{u,v} du\,dv$

$$\{(x,y) \in \mathbb{R}^2 : 0 < x, y < 1\}$$

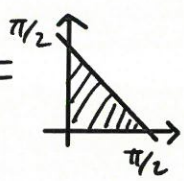
$$\downarrow$$

$$\{(u,v) \in \mathbb{R}^2 ;\ u, v > 0,\ u+v < \pi/2\}$$



③ Conclusion

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \iint_{u,v} du\,dv =$$



$$= \pi^2/8.$$

□

§. Evaluating $S(n)$ by change of variables

$$\Pi_n := \{(u_1, u_2, \dots, u_n) \in \mathbb{R}^n \,;\, u_i > 0,\ u_i + u_{i+1} < \pi/2,\ 1 \le i \le n\},$$

$u_{n+1} := u_1.$

① Let

$$x_i = \frac{\sin u_i}{\cos u_{i+1}},\quad 1 \le i \le n-1, \qquad x_n = \frac{\sin u_n}{\cos u_1}.$$

Then

$$\frac{\partial x_i}{\partial u_j} = \begin{cases} \dfrac{\cos u_i}{\cos u_{i+1}}, & \text{if } j = i, \\ \dfrac{\sin u_i \sin u_{i+1}}{\cos^2 u_{i+1}}, & \text{if } j \equiv i+1 \mod n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\left| \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} \right| = 1 \pm (x_1 \cdots x_n)^2.$$

② Lem. $(0,1)^n \xleftrightarrow{1-1} \Pi_n$.

③
$$\mathrm{Vol}(\Pi_n) = \int \!\cdots\! \int_{\substack{u_i > 0 \\ u_i + u_{i+1} < \pi/2}} 1 \, du_1 \cdots du_n$$

$$= \int_0^1 \!\cdots\! \int_0^1 \frac{dx_1 \cdots dx_n}{1 \pm (x_1 \cdots x_n)^2}$$

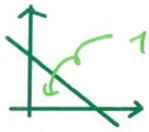
$$= \int_0^1 \!\cdots\! \int_0^1 \sum_{k=0}^{\infty} (-1)^{nk} (x_1 \cdots x_n)^{2k} \, dx_1 \cdots dx_n$$

$$= \sum_{k=0}^{\infty} (-1)^{nk} \int_0^1 \!\cdots\! \int_0^1 (x_1 \cdots x_n)^{2k} \, dx_1 \cdots dx_n$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{nk}}{(2k+1)^n} = S(n).$$

§. Relating $S(n)$ to Π_n via linear operators

Define
$$K_1(u,v) = \begin{cases} 1 & \text{if } \quad u+v < \pi/2 \\ 0 & \text{otherwise}. \end{cases}$$



Then the volume of Π_n can be written as
$$\int_0^{\pi/2} \cdots \int_0^{\pi/2} \prod_{i=1}^{n} K_1(u_i, u_{i+1})\, du_1 \cdots du_n$$
$$= \int_0^{\pi/2} K_n(u,u)\, du, \quad \text{where}$$
$$K_n(u,v) = \int_0^{\pi/2} K_1(u,v)\, K_{n-1}(u_1, v)\, du_1$$
$$= \int_0^{\pi/2} \cdots \int_0^{\pi/2} K_1(u,u_1) \cdot \prod_{i=1}^{n-2} K_1(u_i, u_{i+1}) \cdot K_1(u_{n-1}, v)\, du_1 \cdots du_{n-1}.$$

We interpret K_n and the integral $\int_0^{\pi/2} K_n(u,u)\, du$ in terms of linear operators on $L^2(0, \pi/2)$.

Let T be the linear operator with kernel $K_1(\cdot,\cdot)$ on $L^2(0,\pi/2)$;
$$(Tf)(v) = \int_0^{\pi/2} f(u) K_1(u,v)\, du = \int_0^{\pi/2 - v} f(u)\, du.$$

Then $K_n(\cdot,\cdot)$ is the kernel of T^n :
$$(T^n f)(v) = \int_0^{\pi/2} f(u) K_n(u,v)\, du.$$

Lem. T^n is a compact, self-adjoint operator on $L^2(0,\pi/2)$. It's eigenvalues, each of multiplicity one, are $1/(4k+1)^n$ $(k\in\mathbb{Z})$, with corresponding eigenfunctions $\cos((4k+1)u)$.

Cor. (Orthogonal expansion of an arbitrary L^2-ftn)
For any $f\in L^2(0,\pi/2)$,
$$f=\sum_{k=-\infty}^{\infty} f_k\cos((4k+1)u),$$
with coefficients f_k
$$f_k=\frac{4}{\pi}\int_0^{\pi/2} f(u)\cos((4k+1)u)\,du.$$

$\Rightarrow$ For each $n\geq 0$, we have
$$\int_0^{\pi/2} f(u)(T^nf)(u)\,du=\frac{\pi}{4}\sum_{k=-\infty}^{\infty}\frac{f_k^2}{(4k+1)^n}.$$

§. Transfer operator method of Elkies

Remark.

Lem. (Stanley, '1986)
The volume of the order polytope associated to any partial order $\prec$ on $[n]$ is $1/n!$ times the number of permutations σ of $[n]$ such that $\sigma(i)<\sigma(j)$ whenever $i\prec j$.

Consider
$$P_n=\{(x_1,\dots,x_n)\in[0,1]^n \,;\, x_1\leq x_2\geq x_3\leq\cdots\}.$$
Then, $\boxed{\mathrm{Vol}(P_n)=\frac{1}{n!}E_n.}$

$$\mathrm{Vol}(P_n) = \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \cdots$$

change of variables ; $x_i = \begin{cases} v_i & , i \text{ odd} \\ 1-v_i & , i \text{ even} \end{cases}$

$\Rightarrow \quad x_{i-1} \leq \overset{\text{even}}{x_i} \geq x_{i+1}$

$\Rightarrow \quad v_{i-1} \leq 1 - v_i \geq v_{i+1}$

$\Rightarrow \quad \underline{v_i + v_{i-1} \leq 1}, \quad \underline{v_i + v_{i+1} \leq 1}$

P_n transforms into the region

$\{(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n ; \ v_i \geq 0 , \ v_i + v_{i+1} \leq 1 \ (1 \leq i \leq n-1)\}$.

Further change of variables $v_i = (2/\pi) u_i$;

$\{(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n ; \ u_i \geq 0, \ u_i + u_{i+1} \leq \pi/2 \ (1 \leq i \leq n-1)\}$.

--- $(*)$

Thm. $E_n = \dfrac{2^{n+2}}{\pi^{n+1}} \cdot n! \, S(n+1)$.

pf) ① $E_n = n! \, \mathrm{Vol}(P_n)$

② $\mathrm{Vol}(P_n) = \left(\frac{2}{\pi}\right)^n \cdot \mathrm{Vol}((*))$

$$= \left(\frac{2}{\pi}\right)^n \int_0^{\pi/2} \cdots \int_0^{\pi/2} K_{n-1}(u_1, u_n) \, du_1 \cdots du_n$$

$$= \left(\frac{2}{\pi}\right)^n \langle T^{n-1}(1), 1 \rangle \quad \text{in } L^2(0, \pi/2)$$

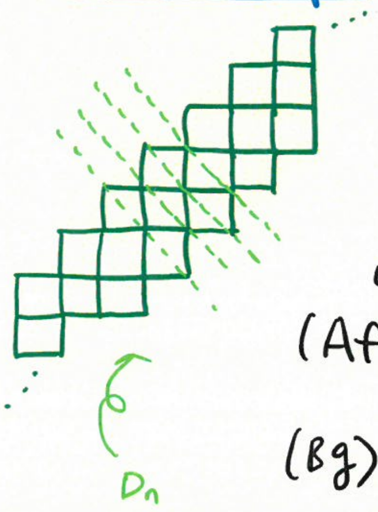
$$= \left(\frac{2}{\pi}\right)^n \cdot \frac{\pi}{4} \sum_{k=-\infty}^{\infty} \frac{c_k^2}{(4k+1)^{n-1}},$$

where $c_k = \frac{4}{\pi} \int_0^{\pi/2} \cos((4k+1)u) \, du = \frac{4}{\pi} \cdot \frac{1}{4k+1}$.

Therefore, $\dfrac{E_n}{n!} = \left(\dfrac{2}{\pi}\right)^n \left(\dfrac{4}{\pi}\right) S(n+1)$. $\square$

$\boxed{3}$ Extension to more complicated shapes

§. 3-strip



$$\Omega = \{(u,v) \in [0,1]^2,\ u < v\}$$

$$A : L_2[0,1] \to L_2(\Omega),$$

$$B : L_2(\Omega) \to L_2[0,1],$$

where

$$(Af)(u,v) = \int_u^v f(x)\,dx,$$

$$(Bg)(x) = \int_0^x \int_x^1 g(u,v)\,dv\,du.$$

Repeat $C := B \circ A$.

Operator method

$$
\begin{aligned}
(Cf)(x) &= (B(Af))(x) \\
&= \int_0^x \int_x^1 (Af)(u,v)\,dv\,du \\
&= \int_0^x \int_x^1 \int_u^v f(y)\,dy\,dv\,du \\
&= \int_0^1 f(y) \left(\int_0^{x\wedge y} du \int_{x\vee y}^1 dv \right) dy \\
&= \int_0^1 f(y)(x\wedge y)(1-x\vee y)\,dy \\
&= \int_0^x f(y)y(1-x)\,dy + \int_x^1 f(y)x(1-y)\,dy \\
&= (1-x)\int_0^x f(y)y\,dy + x\int_x^1 f(y)(1-y)\,dy.
\end{aligned}
$$

It's not too hard to find...

- eigenfunctions ; $\phi_k(x) = \sqrt{2}\sin(\pi k x)$,

& corresponding eigenvalues $\lambda_k = \dfrac{1}{\pi^2 k^2}$, $k = 1,2,3,\dots$

associated polytope to D_n

$$f^{D_n} = (3n)!\, \mathrm{Vol}(P_{D_n})$$; due to Stanley

$$= (3n)! \langle T_{last} \cdot (BA)^{n-1} \cdot T_{first}\, 1, 1 \rangle,$$

where $(T_{first} f)(x) = (T_{last} f)(x) = \int_x^1 f(y)\,dy$

$$= (3n)! \langle C^{n-1}(1-x), x \rangle$$

$$= (3n)! \sum_{k=1}^{\infty} \lambda_k^{n-1} \langle x, \phi_k \rangle \langle 1-x, \phi_k \rangle.$$

(Long computation short...)

$$f^{D_n} = (3n)! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 2}{\pi^2 k^2 (\pi k)^{2(n-1)}}$$

$$= \frac{2(3n)!}{\pi^{2n}} \left(1 - \frac{2}{2^{2n}}\right) \zeta(2n),$$

where $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$.

Bernoulli numbers

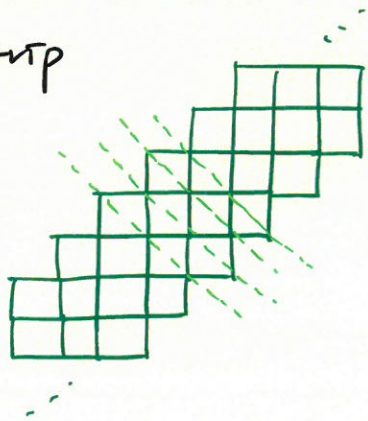
Note. $\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} \pi^{2n} 2^{2n-1} B_{2n}}{(2n)!}$,

where $E_{2n-1} = \frac{(-1)^{n-1} 4^n (4^n - 1)}{2n} B_{2n}$

$$= \frac{(3n)!\,(2^{2n-1} - 1) T_n}{(2n-1)!\, 2^{2n-1} (2^{2n} - 1)}.$$

Further results

- 4-strip



- 2k-strip
- (2k+1)-strip

Things to do...

Can we make q-analogues ??

Thank you