§ Singular values of the Rogers - Ramanujan continued fraction. #1.

§1. Introduction.

$\mathbb{H} :=$ the complex upper half plane

$= \{ \tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0 \}$.

Rogers

$$R(\tau) = q^{1/5} \cdot \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{n}{5}\right)}, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}.$$

Here, $\left(\frac{n}{5}\right)$ denotes the Legendre symbol. R has the expansion

$$R(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + q^3 \cdots}}}$$ as a continued fraction.

Ramanujan $\longrightarrow$ Hardy

$$\cfrac{e^{-\frac{2\pi}{5}}}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + e^{-6\pi} \cdots}}} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

$$\overset{\parallel}{R(i)}$$

$$-R\left(\frac{5+i}{2}\right) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}.$$

$\rightsquigarrow R(\tau)$: a modular function of level 5.

§2. Modular function of level N.

$$SL_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \;\middle|\; ad-bc=1 \right\}.$$

$N \in \mathbb{N}$, $\Gamma(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod N \right\}$.

$(\Rightarrow SL_2(\mathbb{Z}) = \Gamma(1))$

the principal congruence subgroup.

$(\exists N, \Gamma(N) \subset \Gamma \subset \Gamma(1))$.

$\Rightarrow$ For $N \geq 1$, $\Gamma(N)$ acts on $\mathbb{H}$ as a frac. lin. transf.

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \tau \in \mathbb{H} \longmapsto \frac{a\tau+b}{c\tau+d}.$$

$\Rightarrow$ $X(N) := \Gamma(N) \backslash \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ a modular curve of level N.

$\mathbb{C}(X(N)) :=$ the field of all mero. ftns on $X(N)$

$\Rightarrow$ On the plane $\mathbb{H}$, we consider a mero. ftn $f : \mathbb{H} \longrightarrow \mathbb{C}$

s.t. $f\left(\dfrac{a\tau+b}{c\tau+d}\right) = f(\tau)$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(N)$.

& f is mero. on $\mathbb{H}^*$ $\underset{\text{means}}{\overset{= \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}}{}}$ that f has a (Laurent) series expansion

$$f(\tau) = \sum_{n \geq n_0} c_n q^{\frac{n}{N}}, \quad \exists\, n_0 \in \mathbb{Z},\ |n_0| < \infty$$

$$q = e^{2\pi i \tau}$$

$$\mathcal{F}_N := \left\{ f(\tau) = \sum c_n q^{\frac{n}{N}} \in \mathbb{C}(X(N)) \;\middle|\; \forall\, c_n \in \mathbb{Q}\left(e^{\frac{2\pi i}{N}}\right) \right\}.$$

$\Rightarrow \mathcal{F}_N / \mathcal{F}_1$ is a Galois extension.

Here, $F_1 = \mathbb{Q}(j(\tau))$ where

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

is the classical j-invariant in theory of Elliptic Curves.

($SL_2(\mathbb{Z})$ - modular function).

Galois action $\quad Gal(F_N/F_1) \cong GL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \middle| \, d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\} \cdot SL_2(\mathbb{Z}/N\mathbb{Z})/\pm I_2$$

① $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} : f(\tau) = \sum c_n q^{\frac{n}{N}} \longmapsto \sum c_n^{\sigma_d} q^{\frac{n}{N}}$, where

σ_d is the automorphism of $\mathbb{Q}(e^{\frac{2\pi i}{N}})/\mathbb{Q}$ induced

by $\zeta_N^{\sigma_d} = \zeta_N^d$.

② $\gamma \in SL_2(\mathbb{Z}/N\mathbb{Z})/\pm I_2 \; : \; f \longmapsto f \circ \gamma$, a composition

of a frac. lin. transf.

§3. Modularity of $R(\tau)$.

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}$$

Define $\quad h_0(\tau) := \dfrac{\eta(\frac{\tau}{5})}{\eta(\tau)}, \quad h_5(\tau) = \sqrt{5} \cdot \dfrac{\eta(5\tau)}{\eta(\tau)}$.

$$\begin{cases} \dfrac{1}{R(\tau)} - R(\tau) - 1 = \sqrt{5} \cdot \dfrac{h_0(\tau)}{h_5(\tau)} \\[2ex] \dfrac{1}{R(\tau)^5} - R(\tau)^5 - 11 = \dfrac{5^3}{h_5(\tau)^6} \end{cases}$$

(Watson's proof)

$\cdots (*)$

Let $S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be generators of $SL_2(\mathbb{Z})$.

$$\Rightarrow \begin{cases} \eta \circ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}(\tau) = \eta\left(\frac{-1}{\tau}\right) = \sqrt{-i \cdot \tau}\, \eta(\tau) \\ \eta \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}(\tau) = \eta(\tau+1) = e^{\frac{\pi i}{12}} \cdot \eta(\tau). \end{cases}$$

$\Rightarrow$ We can show that $h_5(\tau)^6$, $\dfrac{h_0(\tau)}{h_5(\tau)}$ are modular functions of level 5. (it means they belong to $\mathcal{F}_5$).

$\rightarrow$ By the above fact and $(*)$, we obtain that $R(\tau) \in \mathcal{F}_5$.

※ $R(\tau)$ satisfies a monic irr. poly. of degree 60 in $\mathbb{Z}[j(\tau)][X]$.

§4. Class field theory and complex multiplication.

K : an imag. quad. number field. (e.g. $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, ...)

$\mathcal{O}_K$: the ring of algebraic integers in K.

(the set of elements in K which are roots of monic polynomials in $\mathbb{Z}[X]$).

$\mathfrak{n}$: a nontrivial ideal in $\mathcal{O}_K$.

$I_K(\mathfrak{n})$ = the group of fractional ideals of K prime to $\mathfrak{n}$.

$$P_{K,1}(\mathfrak{n}) = \langle \alpha \mathcal{O}_K \mid \alpha \in \mathcal{O}_K \text{ s.t. } \alpha \equiv 1 \bmod \mathfrak{n} \rangle.$$

$\Rightarrow Cl(\mathfrak{n}) := I_K(\mathfrak{n}) / P_{K,1}(\mathfrak{n})$

the ray class group of K modulus $\mathfrak{n}$.

(Takagi) Existence Theorem

$\exists!$ an abelian extension of K whose Galois group is isomorphic to $Cl(\mathfrak{n})$.

$\Rightarrow$ called the ray class field of K modulo $\mathfrak{n}$, denoted by $K_{\mathfrak{n}}$.

($\mathfrak{n} = \mathcal{O}_K \Rightarrow$ the Hilbert class field, H_K).

Let d_K be a fundamental discriminant d_K.

& $\mathfrak{n} = N\mathcal{O}_K$ for $N \geqslant 1$.

Set

$$\tau_K = \begin{cases} \dfrac{-1+\sqrt{d_K}}{2} & \text{if } d_K \equiv 1 \ (4) \\ \dfrac{\sqrt{d_K}}{2} & \text{if } d_K \equiv 0 \ (4). \end{cases}$$

$\rightarrow$ By CM theory, we get

$$H_K = K(j(\tau_K))$$

$$K_{\mathfrak{n}} = K(h(\tau_K) \mid h(\tau) \in \mathcal{F}_N \text{ finite at } \tau_K).$$

$\Rightarrow$ We have $R(\tau_K) \in K_{5\mathcal{O}_K}$.

※ $R(\tau_K)$ satisfies a monic poly. of degree 60 in $\mathbb{Z}[j(\tau_K)][x]$ and it is well known that $j(\tau_K)$ is an alg. integer. $\Rightarrow R(\tau_K)$ is an alg. integer.

Let $\min(\tau_K, \mathbb{Q}) = X^2 + bX + c$. &

$$W_{K,N} := \left\{ \begin{bmatrix} t-bs & -cs \\ s & t \end{bmatrix} \in GL_2(\mathbb{Z}/N\mathbb{Z}) \;\middle|\; t,s \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

⇒ We have a surjection

$$W_{K,N} \longrightarrow \mathrm{Gal}(K_n/H_K)$$

$$\gamma \longmapsto \left(h(\tau_K) \longmapsto h^{\gamma}(\tau_K) \;\middle|\; h(\tau) \in \mathcal{F}_N,\ \text{finite at } \tau_K \right)$$

& an isomorphism

$$C(d_K) \xrightarrow{\ \sim\ } \mathrm{Gal}(H_K/K)$$

$$\begin{tikzcd} K_{NO_K} \arrow[d, dash, "W_{K,N}/\mathrm{Ker}\sim"] \\ H_K \arrow[d, dash, "C(d_K)"] \\ K \end{tikzcd}$$

⇒ We can show that K_{5O_K} can be generated by $R(\tau_K)$ over K.
(or, $R(\tau_K)$ is a primitive element of K_{5O_K}).

This work is very related to the Hilbert's 12th problem :

Let $\left(\begin{array}{l} K : \text{a finite field ext. of } \mathbb{Q}. \\ L : \text{arbitrary finite abelian ext. of } K. \end{array} \right.$

Is there a nice function $f(\tau)$ s.t. $L = K(f(\alpha))$ for some $\alpha \in \mathbb{C}$?

which is a kind of generalization of

Kronecker - Weber Theorem

L : a finite abelian extension of $\mathbb{Q}$.

$\Rightarrow L \subseteq \mathbb{Q}(e^{\frac{2\pi i}{N}}) \quad \exists$ a positive int. N. //

Ramanujan's cubic continued fraction.

$$C(\tau) = \cfrac{q^{1/3}}{1 + \cfrac{q+q^2}{1 + \cfrac{q^2+q^4}{1 + \cfrac{q^3+q^6}{1+\cdots}}}}$$

$$= q^{1/3} \cdot \prod_{n=1}^{\infty} \frac{(1-q^{6n-1})(1-q^{6n-5})}{(1-q^{6n-3})^2}$$

$\Rightarrow \tau_0 \in \mathbb{H}$ an imag. quadratic

$\Rightarrow C^{-1}(\tau_0)$ is an alg. integer.

$\mathbb{Z}[j(\tau_0)][x]$

& it satisfies a monic poly of deg 4 in $\mathbb{Z}[j][x]$.

$\Rightarrow$ If we know the actual value $j(\tau_0)$ $\exists \tau_0 \in \mathbb{H}$,

then we can express $C(\tau_0)$ in terms of radicals.

$\Downarrow$

monic 이 아님.