

Asymmetric Contests with Initial Probabilities of Winning

By Kyung Hwan Baik and Hanjoon Michael Jung*

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Abstract

We study contests in which players compete with one another by expending irreversible effort to win a prize, and each player has an initial probability of winning the prize. First, we consider a model in which the impact parameter is exogenous. We find that neither the equilibrium number of active players nor their identities nor the equilibrium effort levels of the players depend on the players' initial probabilities of winning. We find also that the possibility that the winner is determined by the players' initial probabilities of winning reduces prize dissipation, and tends to make most players better off, compared to the contest without this possibility. Then, we consider an extended model in which the impact parameter is endogenous. Interestingly, we find that every player may expend zero effort in equilibrium.

JEL classification: D72, C72

Keywords: Asymmetric contest; Initial probability of winning; No-effort equilibrium; Prize dissipation; More efficient rent seeking

*Baik: Department of Economics, Sungkyunkwan University, Seoul 03063, South Korea (e-mail: khbaik@skku.edu); Jung (corresponding author): Ma Yinchu School of Economics, Tianjin University, Tianjin 300072, China (e-mail: hj.jung@tju.edu.cn). We are grateful to Sang-Kee Kim, Amy Baik Lee, Dan Kovenock, Wooyoung Lim, Eric Rasmusen, Jiabin Wu, two anonymous referees, an associate editor and the managing editor of the journal, and seminar participants at Academia Sinica for their helpful comments and suggestions. Earlier versions of this paper were presented at the 13th Annual Conference of the Association for Public Economic Theory, Taipei, Taiwan, June 2012; and the 88th Annual Conference of the Western Economic Association International, Seattle, WA, July 2013. Hanjoon Michael Jung gratefully acknowledges the research support from the National Science Council of the Republic of China (MOST 101-2410-H-001-014).

1. Introduction

We study situations in which players are vested with their initial probabilities of winning a prize, and compete with one another by expending irreversible effort to win the prize. Such contests with initial probabilities of winning are readily observed, but to the best of our knowledge, have not previously been studied.¹ Each player's initial probability of winning may reflect customs, institutional rules, her natural talent, educational background, or previous efforts in the contest or other related contests. For example, in a rent-seeking contest, it may reflect her relationship with government officials established by her previous rent-seeking efforts. In an R&D contest, it may reflect her relevant knowledge and experience gained from her previous R&D efforts. Finally, in a sporting contest, it may reflect her natural talent or sporting skills (or ability) gained from her previous sporting efforts.

We formally consider the following simultaneous-move game. At the beginning of the game, the players know their valuations for the prize, their initial probabilities of winning the prize, and the information regarding the value of an "impact parameter" – the parameter which determines how much impact the players' initial probabilities of winning have on their final probabilities of winning. Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the prize is awarded to one of the players.

We first consider a model in which the impact parameter is exogenous. Then, we consider an extended model in which the impact parameter is endogenous. In these models, the simplest logit-form function is used in specifying each player's contest success function, which is a rule that describes the relationship among her initial probability of winning, the players' effort levels, and her final probability of winning. In the models, the players are assumed to have different valuations for the prize and different initial probabilities of winning the prize. In the extended model, the impact parameter is a nonincreasing function of the total effort level of the players.

In the first model, we show the following results. First, the equilibrium number of active players – that is, the number of players who expend positive effort in equilibrium – and their

identities depend neither on the players' initial probabilities of winning the prize nor on the impact parameter. Second, the equilibrium effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, the possibility that the winner is determined by the players' initial probabilities of winning – in other words, the possibility that the players' efforts are wasted – reduces prize dissipation (or rent dissipation in a rent-seeking contest), compared to the contest without this possibility.² Interestingly, prize dissipation decreases only because of that possibility, but not because of the players' initial probabilities of winning per se. Third, the possibility that the winner is determined by the players' initial probabilities of winning tends to make better off most players with positive initial probability of winning, compared to the contest without this possibility.

In the extended model, we show that every player may expend zero effort in equilibrium. This result contrasts with the received result – that the no-effort equilibrium does not exist – in the literature on the theory of contests. To the best of our knowledge, the existence of the no-effort equilibrium is shown only in Baik (1998), which studies contests with difference-form contest success functions. Another interesting result in the extended model is that there may exist multiple pure-strategy Nash equilibria even though the simplest logit-form function is used in specifying each player's contest success function. This result is interesting because it is well known in the literature on the theory of contests that the Nash equilibrium is unique in contests with the simplest logit-form functions: See, for example, Cornes and Hartley (2005) and Yamazaki (2008).

This paper is closely related to Hillman and Riley (1989). They study both transfer and rent-seeking contests in which the players have different valuations for the prize. We extend their model by incorporating the players' initial probabilities of winning.

This paper is related to Ansink (2011), Faith et al. (2008), and McBride et al. (2011). Ansink (2011) studies resource contests in which the players have exogenously given claims on shares of a contested resource; they compete with one another by exerting effort to secure larger shares of the resource; and their claims and effort levels jointly determine, through a contest-

bankruptcy rule, their final shares of the resource. Faith et al. (2008) study intergenerational-transfer contests in which the parents first choose and commit to their children's bequest shares, and then given their "claims" on bequest shares, the children compete with each other by expending effort to secure larger amounts of bequests and gifts. McBride et al. (2011) study a model of conflict between two parties in which two adversaries first undertake their one-time investments in state capacity capital, and then after observing their levels of state capacity, the two adversaries decide how much to arm and whether to engage in conflict.

This paper is also related to another strand of literature which studies contests with biases, handicaps, head starts, and/or affirmative actions. Examples include Franke (2012), Kirkegaard (2012), Seel and Wasser (2014), and Siegel (2014).

Tullock (1980) and Hillman and Riley (1989) study rent-seeking contests in which the players expend their effort simultaneously, and establish that "efficient rent seeking" occurs – that is, rent dissipation is less than complete. Subsequently, Baik and Shogren (1992) and Leininger (1993) study endogenous timing in lopsided contests, and show that the endogenous timing of moves leads to "more efficient rent seeking" because, in equilibrium, the underdog moves first and restrains herself in order to avoid stiff competition against the favorite, which in turn allows the favorite to ease up and respond efficiently. By contrast, the present paper shows that the presence of the players' initial probabilities of winning, or rather the possibility that the winner is determined by the players' initial probabilities of winning, leads to more efficient rent seeking even when the players move simultaneously.

There exist many papers that study asymmetric contests: See, for example, Morgan (2003), Baik (2004), Stein and Rapoport (2004), Malueg and Yates (2005), Cornes and Hartley (2005), Siegel (2010), and Franke et al. (2013). A striking difference between this paper and the previous ones is that this paper incorporates the players' initial probabilities of winning into their contest success functions, whereas the previous papers do not.

The paper proceeds as follows. Section 2 develops the model in which the impact parameter is exogenous. In Section 3, we solve for the Nash equilibrium of the game, and obtain

the equilibrium number of active players and their identities, the equilibrium effort levels of the players, and the equilibrium expected payoffs for the players. In Section 4, we examine the effects of changing the parameters on these outcomes of the game. Section 5 considers the extended model. Finally, Section 6 offers our conclusions.

2. The model with the simplest logit-form function

Consider a contest in which each of n risk-neutral players, 1 through n , wants to win a prize, and has an initial probability of winning the prize, where $n \geq 2$. The players compete with one another by expending irreversible effort to win the prize. The (final) probability that a player wins the prize is increasing in her own effort level, *ceteris paribus*, and decreasing in the rivals' total effort level. The prize is awarded to one of the players.

Let v_i , for $i = 1, \dots, n$, represent player i 's valuation for the prize. Without loss of generality, we assume that $v_1 \geq v_2 \geq \dots \geq v_n > 0$. Let α_i represent player i 's initial probability of winning the prize, so that $\alpha_i \geq 0$ and $\sum_{j=1}^n \alpha_j = 1$.³ We assume that each player's valuation and her initial probability of winning the prize are publicly known.

Let x_i represent the effort level expended by player i , and let X represent the effort level expended by all the players, so that $X \equiv \sum_{j=1}^n x_j$. Let \mathbf{x} denote an n -tuple vector of effort levels, one for each player: $\mathbf{x} \equiv (x_1, \dots, x_n)$. Each player's effort level is nonnegative, and is measured in units commensurate with the prize. Let p_i denote the (final) probability that player i wins the prize. We assume that each player's (final) probability of winning depends on her initial probability of winning and the players' effort levels. More specifically, we assume the following contest success function for player i :

$$p_i = \theta \alpha_i + (1 - \theta) f_i(\mathbf{x}), \quad (1)$$

where $0 < \theta < 1$, $f_i(\mathbf{x}) = x_i/X$ if $X > 0$, and $f_i(\mathbf{x}) = 1/n$ if $X = 0$. This contest success function says that, *ceteris paribus*, player i 's probability of winning is increasing in her initial probability

of winning. It says also that, *ceteris paribus*, player i 's probability of winning is increasing in her effort level at a decreasing rate; however, it is decreasing in the rivals' total effort level at a decreasing rate. The contest success function implies that player i 's probability of winning may be positive even though she expends zero effort. One interpretation of (1) is that player i 's probability of winning is determined only by her initial probability of winning with probability θ , and is determined only by the value of $f_i(\mathbf{x})$ with probability $(1-\theta)$. Another interpretation is that player i 's probability of winning is determined by a weighted average of her initial probability of winning and the value of $f_i(\mathbf{x})$, where the weights are given by θ and $(1-\theta)$. We assume that the impact parameter θ is exogenous and its value is publicly known.

Stein (2002) considers an n -player contest in which the contest success function for player i , for $i = 1, \dots, n$, is given by $p_i = \lambda_i x_i / \sum_{k=1}^n \lambda_k x_k$, where $0 < \lambda_i \leq 1$. He argues that the parameter λ_i can be interpreted as the rate at which player i 's effort level is converted into her effective effort level; alternatively, it may be interpreted as a measure of player i 's prior relative chance of winning. A striking difference between the specification in (1) and that in Stein (2002) is that (1) is additively separable in player i 's initial probability of winning and the players' effort levels, whereas player i 's contest success function in Stein (2002) is not. As a result, in Stein (2002), the specification of player i 's contest success function may reflect her initial probability of winning only if her effort level is positive. In other words, given a positive effort level from the other players, player i 's probability of winning is zero if her effort level is zero, regardless of how high her initial probability of winning is. However, this is not the case with the specification in (1) in this paper. Accordingly, we believe that contests in which players are vested with their initial probabilities of winning can be better modeled with the specification in (1) in this paper.

We formally consider the following noncooperative simultaneous-move game. At the beginning of the game, the players know their valuations for the prize, their initial probabilities

of winning the prize, and the value of θ . Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the winner is determined.

Let π_i represent the expected payoff for player i . Then the payoff function for player i is

$$\pi_i = v_i\{\theta\alpha_i + (1 - \theta)f_i(\mathbf{x})\} - x_i. \quad (2)$$

Recall that, after choosing an effort level of x_i , player i earns a payoff of v_i if she wins the prize and 0 if she loses it, and that her probability of winning the prize is $\theta\alpha_i + (1 - \theta)f_i(\mathbf{x})$.

We assume that all of the above is common knowledge among the players. We employ Nash equilibrium as the solution concept.

3. Nash equilibrium: Who are the active players?

To obtain a Nash equilibrium of the game, we begin by considering the following maximization problem facing player i for $i = 1, \dots, n$: Maximize player i 's expected payoff π_i in (2) over her effort level, $x_i \geq 0$, given effort levels of all the other players. We obtain Lemma 1.

Lemma 1. (a) *The strategy profile, $\mathbf{x} = (0, \dots, 0)$, at which every player expends zero effort does not constitute a Nash equilibrium.* (b) *A strategy profile at which only one player expends positive effort does not constitute a Nash equilibrium.*

The proof of Lemma 1 is straightforward. To prove part (a), we use the fact that each player has an incentive to increase her effort level from zero to an infinitesimally small positive, ϵ , given zero effort levels of the other players. To prove part (b), we use the fact that the player who expends positive effort has an incentive to decrease her effort level, given zero effort levels of the other players.

Let \mathbf{x}_{-i} denote an $(n-1)$ -tuple vector of effort levels, one for each player except player i : $\mathbf{x}_{-i} \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Let $X_{-i} \equiv \sum_{j \neq i} x_j$. Utilizing Lemma 1, to obtain a Nash

equilibrium of the game, it suffices to consider the following simplified maximization problem facing player i : Given a vector of effort levels, \mathbf{x}_{-i} , such that $X_{-i} > 0$,

$$\text{Max}_{x_i} \pi_i = v_i \{ \theta \alpha_i + (1 - \theta)(x_i/X) \} - x_i \quad (3)$$

subject to the nonnegativity constraint, $x_i \geq 0$.

Let x_i^b denote player i 's best response to \mathbf{x}_{-i} . Then, by definition, it satisfies the first-order condition:

$$\{v_i(1 - \theta)X_{-i}/(X_{-i} + x_i^b)^2\} - 1 = 0 \quad \text{for } x_i^b > 0 \quad (4)$$

or

$$\{v_i(1 - \theta)X_{-i}/(X_{-i} + x_i^b)^2\} - 1 \leq 0 \quad \text{for } x_i^b = 0. \quad (5)$$

It is straightforward to see that the objective function in (3) is strictly concave in x_i , which implies that the second-order condition is satisfied and x_i^b is unique.

Let \mathbf{x}^* denote a Nash equilibrium: $\mathbf{x}^* \equiv (x_1^*, \dots, x_n^*)$. Let $X^* \equiv \sum_{j=1}^n x_j^*$. At the Nash equilibrium, player i 's effort level is the best response to the other players' effort levels. Thus it satisfies either (6) or (7), given the other players' equilibrium effort levels:

$$\{v_i(1 - \theta)(X^* - x_i^*)/(X^*)^2\} - 1 = 0 \quad \text{for } x_i^* > 0 \quad (6)$$

or

$$\{v_i(1 - \theta)(X^* - x_i^*)/(X^*)^2\} - 1 \leq 0 \quad \text{for } x_i^* = 0. \quad (7)$$

Note that (6) and (7) come from (4) and (5), respectively. Using (6) and (7), it is straightforward to obtain Lemma 2, which compares the players' equilibrium effort levels.

Lemma 2. *At a Nash equilibrium, any two players with the same valuation for the prize expend the same effort. The effort level of a player with a higher valuation, if positive, is greater than*

that of a player with a lower valuation; otherwise, it is equal to that of a player with a lower valuation. Thus $x_{k-1}^* \geq x_k^*$ holds for $k = 2, \dots, n$.

Lemma 2 follows from the following facts. The first is that, *ceteris paribus*, the marginal gross payoff (of increasing effort) of a higher-valuation player is greater than that of a lower-valuation player at any effort level. The second is that each player's marginal gross payoff is decreasing in her own effort level. The third fact is that the marginal cost of increasing effort is constant and the same for all the players. Lemma 2 says that the equilibrium effort level of a highest-valuation player is the largest, that of a lowest-valuation player is the smallest, and the rest are in between.

We are now prepared to obtain the Nash equilibrium of the game. Let n^* denote the number of the players who expend positive effort at the Nash equilibrium. Then it follows from Lemma 1 that $2 \leq n^* \leq n$. It follows from Lemma 2 that players 1 through n^* expend positive effort, and the rest, if any, expend zero effort. Using (6) and (7), we obtain the Nash equilibrium (see Appendix A).

Proposition 1. (a) *The following strategy profile constitutes the Nash equilibrium of the game.*

Player i , for $i = 1, \dots, n^$, plays the strategy $x_i^* = (1 - \theta)(n^* - 1)\{v_i \sum_{j=1}^{n^*} (1/v_j) - n^* + 1\} / v_i \{\sum_{j=1}^{n^*} (1/v_j)\}^2$, where $v_{n^*} \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 > 0$. Player k , for $k = n^* + 1, \dots, n$, if any, expends zero effort, in which case $v_k \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 \leq 0$ holds.⁴ (b) *At the Nash equilibrium, the total effort level of the players is $X^* = (1 - \theta)(n^* - 1) / \sum_{j=1}^{n^*} (1/v_j)$.**

Under the assumption that $v_1 \geq v_2 \geq \dots \geq v_n > 0$, part (a) of Proposition 1 implies that $v_i \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 > 0$ and $v_i \sum_{j=1}^i (1/v_j) - i + 1 > 0$ hold for $i = 1, \dots, n^*$.⁵ Note that the top

two players (according to valuation for the prize) are *always* active in equilibrium – in other words, players 1 and 2 expend positive effort, regardless of the number of players, their valuations for the prize, their initial probabilities of winning the prize, and the value of θ .

Proposition 1 implies the following results. First, the equilibrium number of active players and their identities are determined only by the players' valuations. Indeed, they depend neither on the players' initial probabilities of winning the prize nor on the impact parameter θ . This can be explained as follows. Rewriting (2) as $\pi_i = v_i\theta\alpha_i + \{v_i(1 - \theta)f_i(\mathbf{x}) - x_i\}$, we see that the equilibrium effort levels of the players are equal to those obtained in a reduced contest in which (i) the same n players compete to win the prize in the absence of initial probabilities of winning and (ii) their valuations are reduced proportionately by a factor of $(1 - \theta)$. Accordingly, the equilibrium number of active players and their identities do not depend on the players' initial probabilities of winning. They do not depend on the impact parameter θ , either, because any proportionate decrease in the players' valuations do not affect the positivity and nonpositivity conditions in Proposition 1, which are used to identify the active or inactive players. Second, the equilibrium effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, they do depend on the impact parameter θ . Indeed, the possibility that the winner is determined by the players' initial probabilities of winning – in other words, any positive value of the impact parameter θ – reduces each active player's equilibrium effort level and thus the equilibrium total effort level, compared to the contest without this possibility.⁶ All these follow from two facts. The first is that the equilibrium effort levels of the players are equal to those obtained in a reduced contest in which the same n players compete to win the prize in the absence of initial probabilities of winning and their valuations are reduced proportionately by a factor of $(1 - \theta)$. The second fact is that any proportionate decrease in the players' valuations decreases the equilibrium effort levels of the active players proportionately. More specifically, a decrease in each active player's valuation decreases her marginal gross payoff at every effort level while her marginal cost remains unchanged, so that her equilibrium effort level decreases. Third, in equilibrium, players who have relatively low valuations may not

be active. Fourth, the equilibrium effort level of each active player, and thus the equilibrium total effort level, depends on the valuations of the active players, but not directly on the valuations of the inactive players. Fifth, in equilibrium, a player has a positive (final) probability of winning and may be selected as the winner, unless her initial probability of winning is zero or she expends zero effort. This implies that the winner may not be an active player.

If we assume that $v_1 \geq v_j = v$ for $j = 2, \dots, n$, Proposition 1 is reduced to Corollary 1.

Corollary 1. *In the case where $v_1 \geq v_j = v$ for $j = 2, \dots, n$, we have (a) $n^* = n$ at the Nash equilibrium, (b) $x_1^* = v_1 v (1 - \theta)(n - 1) \{ (n - 1)v_1 - (n - 2)v \} / \{ (n - 1)v_1 + v \}^2$, (c) $x_j^* = v_1 v^2 (1 - \theta)(n - 1) / \{ (n - 1)v_1 + v \}^2$, and (d) $X^* = v_1 v (1 - \theta)(n - 1) / \{ (n - 1)v_1 + v \}$.*

Corollary 1 says that all the players are active in equilibrium, regardless of the size of v_1 , if players 2 through n have the same valuation. This can be explained as follows. According to Proposition 1, in equilibrium, player 1's decision on her effort level always leaves some room for a second-ranked player (according to valuation for the prize) to expend positive effort. If players 2 through n have the same valuation, then the $(n - 1)$ players all are second-ranked players, and thus expend the same positive effort. An interesting implication of this result is that, facing one giant rival contestant, the rest never give up, no matter how small each may be, if they are equal.

If we assume that $v_i = v$ for all $i = 1, \dots, n$, then the players expend the same effort even though the players' initial probabilities of winning differ. Note that, in this case, the equilibrium total effort level is less than the players' common valuation v – in other words, underdissipation of the prize occurs.

Next, we look at the expected payoffs for the players at the Nash equilibrium. Let π_i^* represent the equilibrium expected payoff for player i for $i = 1, \dots, n$. Using (2) and Proposition 1, we obtain Proposition 2 and Corollary 2.

Proposition 2. *At the Nash equilibrium, the expected payoff for player i is equal to*

$$\pi_i^* = v_i \theta \alpha_i + (1 - \theta) \left\{ v_i \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 \right\}^2 / v_i \left\{ \sum_{j=1}^{n^*} (1/v_j) \right\}^2 \text{ for } i = 1, \dots, n^*, \text{ and}$$

$$\pi_i^* = v_i \theta \alpha_i \text{ for } i = n^* + 1, \dots, n.$$

Note that the second term in the right-hand side of the first expression is positive for player i for $i = 1, \dots, n^*$. This says that each active player's *net* expected payoff from expending effort is positive.

Proposition 2 says that the equilibrium expected payoff for each player depends on her own initial probability of winning the prize, the impact parameter θ , and the equilibrium number of active players, as well as the players' valuations for the prize. Thus our assumption that $v_1 \geq v_2 \geq \dots \geq v_n > 0$ may not lead to the result that $\pi_{k-1}^* \geq \pi_k^*$ for $k = 2, \dots, n$. Next, consider any two players with the same initial probability of winning the prize. Proposition 2 together with Lemma 2 implies that the equilibrium expected payoff for a player with a higher valuation, if positive, is greater than that of a player with a lower valuation.

Corollary 2. *In the case where $v_1 \geq v_j = v$ for $j = 2, \dots, n$, we have $\pi_1^* = v_1 \theta \alpha_1 + v_1 (1 - \theta) \{(n-1)v_1 - (n-2)v\}^2 / \{(n-1)v_1 + v\}^2$ and $\pi_j^* = v \theta \alpha_j + v^3 (1 - \theta) / \{(n-1)v_1 + v\}^2$.*

If we assume that $v_i = v$ for all $i = 1, \dots, n$, then the equilibrium expected payoff for a player increases, compared to the contest without initial probabilities of winning the prize, if her initial probability of winning the prize is greater than $1/n^2$. A player with a higher initial probability of winning the prize has a greater equilibrium expected payoff than a player with a lower initial probability of winning the prize.

4. The effects of changing the parameters

We first examine how the outcomes of the game – specifically, the equilibrium number of active players and their identities, the equilibrium effort levels of the players, and the equilibrium expected payoffs for the players – respond when the players' initial probabilities of winning change. Proposition 3 is immediate from Propositions 1 and 2.

Proposition 3. *If the players' initial probabilities of winning the prize change, then (a) the equilibrium number n^* of active players and their identities remain unchanged, (b) the equilibrium effort level of each player, and thus the equilibrium total effort level, remains unchanged, and (c) the equilibrium expected payoff for each player changes in the same direction as her initial probability of winning the prize does.*

Proposition 3 says that changes in the players' initial probabilities of winning do not affect the equilibrium number of active players, their identities, or the equilibrium effort levels of the players, but do affect the equilibrium expected payoffs for the players.

Next, we examine the effects of increasing the impact parameter θ , *ceteris paribus*, on the outcomes of the game. Using Propositions 1 and 2, we obtain Proposition 4.

Proposition 4. *As the impact parameter θ increases, (a) the equilibrium number of active players and their identities remain unchanged, (b) the equilibrium effort level of each active player, and thus the equilibrium total effort level, decreases, (c) the equilibrium expected payoff for an inactive player, if any, whose initial probability of winning the prize is positive, increases, and (d) the equilibrium expected payoff for player i , for $i = 1, \dots, n^*$, increases if $\alpha_i > K_i$, remains unchanged if $\alpha_i = K_i$, and decreases if $\alpha_i < K_i$, where $K_i \equiv \{1 - (n^* - 1)(1/v_i) / \sum_{j=1}^{n^*} (1/v_j)\}^2$.*

If the impact parameter θ increases, then, according to (1), player i 's probability of winning is determined to be such that more weight is given to her initial probability of winning and less weight is given to the value of $f_i(x)$. Consequently, each active player is less motivated to expend effort, and actually expends less effort than before the change. The following complement part (d) of Proposition 4. First, part (d) implies that an increase in θ is likely to increase the equilibrium expected payoff for an active player if her initial probability of winning is "large." Second, it implies that, because $\partial K_i / \partial v_i > 0$, an increase in θ is likely to increase the equilibrium expected payoff for player i if her valuation for the prize is "small." Third, it implies that, because $\partial K_i / \partial v_j < 0$ for $j = 1, \dots, n^*$ with $j \neq i$, an increase in θ is likely to increase the equilibrium expected payoff for player i if player j 's valuation is "large." Fourth, in the case where there are just two players, it is easy to see that K_i in part (d), for $i = 1, 2$, is reduced to $v_i^2 / (v_1 + v_2)^2$.

Finally, we examine the effects of changing the players' valuations for the prize on the outcomes of the game. Using Propositions 1 and 2, we obtain Proposition 5.

Proposition 5. *If the valuations of one or more active players increase, then the equilibrium number of active players never increases, but may decrease. If the valuations of one or more inactive players decrease, then (a) the equilibrium number of active players and their identities remain unchanged, (b) the equilibrium effort level of each player, and thus the equilibrium total effort level, remains unchanged, and (c) their equilibrium expected payoffs, if positive, decrease, while the equilibrium expected payoffs of the rest of the players remain unchanged.*

In addition to Proposition 5, we obtain the following results. First, if the valuations of the active players increase at the same percentage rate, then the equilibrium number of active players and their identities remain unchanged; however, their equilibrium effort levels and their equilibrium expected payoffs, respectively, increase at the same percentage rate, and furthermore increase at the very percentage rate at which their valuations increase. Second, if the valuations

of all the players increase (decrease) proportionately, then the equilibrium number of active players and their identities remain unchanged; however, the equilibrium effort levels of the active players and the equilibrium expected payoffs (if positive) of the players, respectively, increase (decrease) proportionately.

5. An extension: endogenizing the impact parameter θ

The extended model is the same as the model in Section 2 with the exception that, in the extended model, the impact parameter θ is a nonincreasing function of the total effort level X of the players. Interestingly, we show that every player may expend zero effort in equilibrium.

Let the impact parameter θ be a function of the total effort level X of the players: $\theta = \theta(X)$. We assume that the function θ has the following properties.

Assumption 1. *Let θ be a function from R_+ to the unit interval $[0, 1]$, where R_+ denotes the set of all positive real numbers. We assume that $\theta(0) = 1$; $\theta(X) = 0$ for all X with $X \geq M$, where $M \in (0, \infty]$; and $\lim_{X \rightarrow \infty} \theta(X) = 0$. We assume also that the function θ is continuous on R_+ and $\theta'(X) < 0$ for all X with $0 \leq X < M$, where θ' denotes the first derivative of the function θ .*

In Assumption 1, we assume that each player's probability of winning is determined only by her initial probability of winning if the total effort level X of the players is equal to zero. We assume also that, as the total effort level X of the players increases, the impact parameter θ decreases, which implies that player i 's probability of winning is determined to be such that less weight is given to her initial probability of winning and more weight is given to the value of $f_i(\mathbf{x})$. Assumption 1 may reflect the idea that as the total effort level X of the players increases, the decision-maker who has authority to select the winner should value the players' effort more and give less weight to their initial probabilities of winning. One example of the function θ which satisfies the properties in Assumption 1 is $\theta(X) = 1 - \min\{\delta X, V\}/V$, where $\delta > 0$ and

$V \equiv \sum_{j=1}^n v_j$. Another example of the function θ is $\theta(X) = \exp(-\eta X)$, where $\eta > 0$. Note that

$M = V/\delta$ in the first example, and that $M = \infty$ in the second example.

The contest success function for player i is then

$$p_i = \theta(X)\alpha_i + (1 - \theta(X))f_i(\mathbf{x}),$$

and her payoff function is

$$\pi_i = v_i\{\theta(X)\alpha_i + (1 - \theta(X))f_i(\mathbf{x})\} - x_i.$$

Now we show that, under a certain condition, a Nash equilibrium of the game exists in which every player expends zero effort. Let \mathbf{x}^N denote the Nash equilibrium of a game which is the same as the game in Section 2 or this section with the exception that the players compete to win the prize in the absence of initial probabilities of winning, where $\mathbf{x}^N \equiv (x_1^N, \dots, x_n^N)$. Let $X^N \equiv \sum_{j=1}^n x_j^N$. Then we obtain that $X^N = (n^N - 1)/\sum_{j=1}^{n^N} (1/v_j)$, where n^N denotes the number of the players who expend positive effort at the Nash equilibrium \mathbf{x}^N of the game.

Proposition 6. (a) If $\theta'(X)(\alpha_i - 1) \leq 1/v_i$ for all $i = 1, \dots, n$ and for all X with $0 \leq X < M$, then the strategy profile, $\mathbf{x} = (0, \dots, 0)$, at which every player expends zero effort constitutes a Nash equilibrium. In this case, there may exist multiple pure-strategy Nash equilibria. (b) If $\theta'(X)(\alpha_i - x_i/X) + (1 - \theta(X))(X - x_i)/X^2 < 1/v_i$ for all $i = 1, \dots, n$ and for all X with $0 \leq X < M$, and if $X^N \leq M$, then the strategy profile, $\mathbf{x} = (0, \dots, 0)$, at which every player expends zero effort is a unique Nash equilibrium.

The proof of Proposition 6 is provided in Appendix B. Note that, if the specified conditions in part (b) are satisfied, then so is the specified condition in part (a). Proposition 6 says that every player may expend zero effort in equilibrium. This result contrasts with Lemma 1, and further with the received result – that the no-effort equilibrium does not exist – in the

literature on the theory of contests. Proposition 6 says also that there may exist multiple pure-strategy Nash equilibria, which contrasts with the well-known result – that the Nash equilibrium is unique in contests with the simplest logit-form functions – in the literature on the theory of contests.

The existence of the no-effort equilibrium appears counterintuitive because one may expect that player i is more motivated to expend effort if an increase in X gives more weight to the value of $f_i(\mathbf{x})$. However, this appearance is wrong. One should not overlook the fact that, when increasing her effort level, player i not only increases her cost but also decreases the weight to be given to her initial probability of winning. Indeed, the no-effort equilibrium occurs if the condition in part (a) is met, because player i 's marginal gross payoff at every effort level does not exceed her marginal cost given zero effort levels of the other players.

Note that the condition in part (a) is more likely to hold – thus the no-effort equilibrium is more likely to occur – as the players' valuations for the prize decrease. Note also that, if the condition in part (a) is not satisfied, then the no-effort equilibrium does not occur, and at least two players expend positive effort in an equilibrium, if any (see the proof of part (b) in Appendix B).

To better understand Proposition 6, consider a contest in which $n = 2$, $v_1 = v_2 = 1/2$, $\alpha_1 = \alpha_2 = 1/2$, and $\theta(X) = 1 - \min\{\delta X, V\}/V$, where $\delta > 0$ and $V \equiv v_1 + v_2$. If $0 < \delta < 4$, then the strategy profile, $\mathbf{x} = (0, 0)$, is a unique Nash equilibrium. If $\delta = 4$, then the strategy profile, $\mathbf{x} = (0, 0)$, at which both players expend zero effort is not the only Nash equilibrium; indeed, the strategy profiles at which one player expends zero effort and the other player expends an effort level of $1/8$ also are Nash equilibria. If $\delta > 4$, then the condition in part (a) is not satisfied, and thus the strategy profile, $\mathbf{x} = (0, 0)$, is not a Nash equilibrium. In this case, however, there exists a unique Nash equilibrium in which both players expend positive effort – more precisely, each player expends an effort level of $1/8$. Next, considering a contest in which $n = 2$, $v_1 = v_2$, $\alpha_1 = \alpha_2$, and $\theta(X) = \exp(-X)$, we find that the strategy profile, $\mathbf{x} = (0, 0)$, is a unique Nash equilibrium.

6. Conclusions

We have considered the model with the simplest logit-form function in Section 2. Alternatively, we may consider the model with the all-pay-auction selection rule, which is the same as the model in Section 2 with the exception that $f_i(\mathbf{x})$ in (1) is now replaced with $h_i(\mathbf{x})$, where $h_i(\mathbf{x}) = 1/m$ if player i is one of m players expending the largest effort and $h_i(\mathbf{x}) = 0$ if $x_i < x_k$ holds for some k with $k \neq i$. In this alternative model, focusing on the case where $v_1 \geq v_2 > v_3 \geq \dots \geq v_n$, we obtain the following. First, only the top two players according to valuation for the prize are active at the unique mixed-strategy Nash equilibrium. Second, the equilibrium number of active players and their identities depend neither on the players' initial probabilities of winning the prize nor on the impact parameter θ . Third, the equilibrium expected effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, the possibility that the winner is determined by the players' initial probabilities of winning reduces the equilibrium expected effort levels of the two active players, compared to the contest without this possibility. Fourth, except the top two players (according to valuation for the prize), any player can never be the winner if her initial probability of winning the prize is zero. Fifth, a player with a higher initial probability of winning the prize has a greater equilibrium expected payoff than a player with a lower initial probability of winning the prize. Sixth, the equilibrium expected payoff for a player with a higher valuation may be less than that of a player with a lower valuation. Seventh, the possibility that the winner is determined by the players' initial probabilities of winning may make the highest-valuation player worse off but makes better off other players with positive initial probability of winning, compared to the contest without this possibility.

It would be interesting to study an extended model in which the players' initial probabilities of winning the prize are endogenous. In this paper, the winner is selected among the players whose final probabilities of winning the prize are positive. It would be interesting to study a model in which the winner is selected only among the active players. Finally, it would

be interesting to study a model in which the players have incomplete information about the value of the impact parameter θ . We leave these extensions and modifications for future research.

Footnotes

1. A contest is defined as a situation in which players or contestants compete with one another by expending irreversible effort or resources to win a prize. Examples of contests include rent-seeking contests, R&D contests, sporting contests, election campaigns, litigation, tournaments, all-pay auctions, environmental conflicts, and arms races. Important work in the literature on the theory of contests includes Tullock (1980), Dixit (1987), Hillman and Riley (1989), Epstein and Nitzan (2007), and Konrad (2009).

2. By prize dissipation, we mean the equilibrium total effort level, which is "dissipated" in pursuit of the prize. In the literature on the theory of contests, one of the main issues is how much prize dissipation takes place.

3. One may interpret α_i as player i 's initial fractional share of the prize or her initial fractional claim on the prize.

4. Using these positivity and nonpositivity conditions, we identify the active players at the Nash equilibrium. Proposition 5 in Hillman and Riley (1989) states similar conditions.

5. It follows from part (a) of Proposition 1 that $v_{n^*} \sum_{j=1}^{n^*-1} (1/v_j) - (n^* - 1) + 1 > 0$ holds.

Then, under the assumption that $v_1 \geq v_2 \geq \dots \geq v_n > 0$, we have $v_{n^*-1} \sum_{j=1}^{n^*-1} (1/v_j)$

$- (n^* - 1) + 1 > 0$, which in turn implies that $v_{n^*-1} \sum_{j=1}^{n^*-2} (1/v_j) - (n^* - 2) + 1 > 0$ holds. Then,

under the assumption that $v_1 \geq v_2 \geq \dots \geq v_n > 0$, we have $v_{n^*-2} \sum_{j=1}^{n^*-2} (1/v_j) - (n^* - 2) + 1 > 0$,

and so on. Hence, $v_i \sum_{j=1}^i (1/v_j) - i + 1 > 0$ holds for $i = 1, \dots, n^*$.

6. We can obtain the outcomes of the game without initial probabilities of winning by setting θ equal to zero in the outcomes of the present game.

Appendix A: Obtaining the equilibrium effort levels of the n^* active players

From (6), for $i = 1, \dots, n^*$, we have

$$v_i(1 - \theta)(X^* - x_i^*)/(X^*)^2 - 1 = 0,$$

which can be rewritten as

$$(X^*)^2/v_i = (1 - \theta)(X^* - x_i^*). \quad (\text{A1})$$

Adding these equations together, we have

$$\left\{ \sum_{j=1}^{n^*} (1/v_j) \right\} (X^*)^2 = (1 - \theta)(n^* X^* - X^*).$$

This yields

$$X^* = (1 - \theta)(n^* - 1) / \sum_{j=1}^{n^*} (1/v_j).$$

Substituting this expression for X^* into (A1), we obtain the equilibrium effort levels of the n^* active players.

Appendix B: Proof of Proposition 6

(a) Consider the following problem facing player i for $i = 1, \dots, n$: Given zero effort levels of the other players,

$$\begin{aligned} \text{Max}_{x_i} \quad & \pi_i = v_i \{ \theta(X) \alpha_i + (1 - \theta(X)) f_i(\mathbf{x}) \} - x_i \\ \text{subject to} \quad & x_i \geq 0. \end{aligned}$$

Under the given condition that $\theta'(X)(\alpha_i - 1) \leq 1/v_i$ for all X with $0 \leq X < M$, we obtain that $\partial \pi_i / \partial x_i \leq 0$ for all x_i with $0 \leq x_i < M$, and that $\partial \pi_i / \partial x_i = -1$ for all x_i with $x_i \geq M$. This implies that player i has no incentive to deviate from her strategy of expending zero effort.

For the proof of the second claim that there may exist multiple pure-strategy Nash equilibria, we provide an example in the last paragraph of Section 5.

(b) Consider the following problem facing player i for $i = 1, \dots, n$: Given zero effort levels of the other players,

$$\begin{aligned} \text{Max}_{x_i} \quad & \pi_i = v_i \{ \theta(X) \alpha_i + (1 - \theta(X)) f_i(\mathbf{x}) \} - x_i \\ \text{subject to} \quad & x_i \geq 0. \end{aligned}$$

Given zero effort levels of the other players, the first specified condition in part (b) is equivalent to $\theta'(X)(\alpha_i - 1) < 1/v_i$ for all X with $0 \leq X < M$. Under this condition, we obtain that $\partial \pi_i / \partial x_i < 0$ for all x_i with $0 \leq x_i < M$, and that $\partial \pi_i / \partial x_i = -1$ for all x_i with $x_i \geq M$. This implies that player i 's best response is to expend zero effort and is unique. (Note that this in turn implies that there is no Nash equilibrium at which only one player expends positive effort.)

Next, we show that there is no Nash equilibrium at which at least two players expend positive effort. Suppose on the contrary that there is such a Nash equilibrium, denoted by \mathbf{x}^D , where $\mathbf{x}^D \equiv (x_1^D, \dots, x_n^D)$. Let N^D denote the set of the players who expend positive effort at that Nash equilibrium. Now, consider the following problem facing player k for $k \in N^D$: Given the $(n-1)$ -tuple vector \mathbf{x}_{-k}^D of effort levels of the other players,

$$\begin{aligned} \text{Max}_{x_k} \quad & \pi_k = v_k \left[\theta \left(\sum_{j \neq k} x_j^D + x_k \right) \alpha_k + \left\{ 1 - \theta \left(\sum_{j \neq k} x_j^D + x_k \right) \right\} \left\{ x_k / \left(\sum_{j \neq k} x_j^D + x_k \right) \right\} \right] - x_k \\ \text{subject to} \quad & x_k \geq 0. \end{aligned}$$

Let x_k^B denote player k 's best response to \mathbf{x}_{-k}^D . We have two cases to consider: (i) $\sum_{j \neq k} x_j^D + x_k^B \leq M$ for some k with $k \in N^D$ and (ii) $\sum_{j \neq k} x_j^D + x_k^B > M$ for all k with $k \in N^D$. We show that a contradiction arises in each of those two cases.

First, we consider case (i). Under the first specified condition in part (b), we obtain that $\partial \pi_k / \partial x_k < 0$ for all x_k with $0 \leq x_k < M - \sum_{j \neq k} x_j^D$. We obtain also that $\partial \pi_k / \partial x_k$ (or, precisely,

the left-side derivative of π_k) is nonpositive at $x_k = M - \sum_{j \neq k} x_j^D$. This both implies that x_k^B is equal to 0 and is unique, which contradicts that player k expends positive effort, x_k^D , at the Nash equilibrium \mathbf{x}^D .

Second, we consider case (ii). Since $\theta(X) = 0$ for all X with $X > M$ due to Assumption 1, we have

$$\partial \pi_k / \partial x_k = v_k \sum_{j \neq k} x_j^D / (\sum_{j \neq k} x_j^D + x_k)^2 - 1.$$

If $x_k^B > 0$, then, by definition, x_k^B satisfies the following first-order condition:

$$v_k \sum_{j \neq k} x_j^D / (\sum_{j \neq k} x_j^D + x_k^B)^2 = 1. \quad (\text{B1})$$

Since \mathbf{x}^D is a Nash equilibrium, it must satisfy these first-order conditions for all k with $k \in N^D$, and satisfy the condition for case (ii) that $X^D > M$, where $X^D \equiv \sum_{j=1}^n x_j^D$.

Now, consider a game which is the same as the game in Section 2 or Section 5 with the exception that the players compete to win the prize in the absence of initial probabilities of winning. It is well known in the literature on the theory of contests that the Nash equilibrium of the game is unique: See, for example, Cornes and Hartley (2005) and Yamazaki (2008). The Nash equilibrium \mathbf{x}^N of the game must satisfy the first-order conditions – as in (B1) – for the players who expend positive effort.

The game under consideration here in case (ii) can be considered as the one without initial probabilities of winning the prize, mentioned in the preceding paragraph. Thus, in case (ii), X^D must be equal to X^N . This, together with the second condition in part (b) that $X^N \leq M$, yields that $X^D \leq M$. Clearly, this contradicts that $X^D > M$, which is obtained above. \square

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